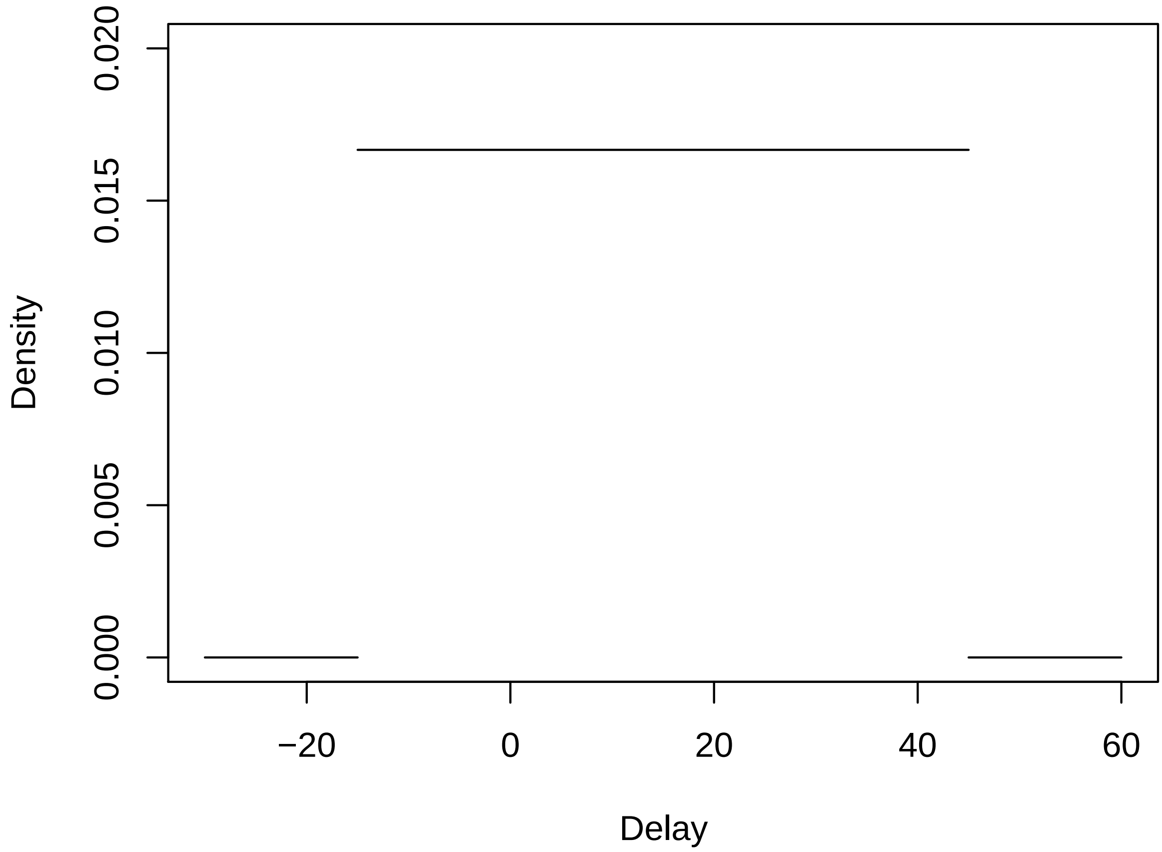
Solutions to Exercises 9

1. The amount of time (in minutes) that the coach is delayed X has a uniform distribution on $a = -15$ to $b = 45$.

   (a) The pdf is a flat line, height $1/\{45 - (-15)\} = 1/60$, in the range -15 to 45. The pdf is zero everywhere else.

   (b) The mean of this distribution is

   $$E(X) = \frac{a+b}{2} = \frac{45 + (-15)}{2} = 15 \text{ minutes},$$

   so that, on average, the coach is 15 minutes late. Also, the variance is

   $$Var(X) = \frac{\{45 - (-15)\}^2}{12} = \frac{3600}{12} = 300$$

   and therefore $SD(X) = \sqrt{Var(X)} = \sqrt{300} = 17.32$ minutes.

(c) Probabilities for this distribution are calculated using

$$P(X \le x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

$$= \begin{cases} 0 & \text{for } x < -15 \\ \dfrac{x+15}{60} & \text{for } -15 \le x \le 45 \\ 1 & \text{for } x > 45. \end{cases}$$

The probability that the coach is less than 5 minutes late is

$$P(X < 5) = \frac{5+15}{60} = \frac{20}{60} = \frac{1}{3} = 0.3333.$$

(d) The probability that the coach is more than 20 minutes late is

$$P(X > 20) = 1 - P(X \le 20) = 1 - \frac{20+15}{60} = 1 - \frac{35}{60} = \frac{5}{12} = 0.4167.$$

(e) The probability that the coach arrives between 22.55 and 23.20 is

$$\begin{aligned} P(-5 < X < 20) &= P(X < 20) - P(X \le -5) \\ &= \frac{20+15}{60} - \frac{-5+15}{60} \\ &= \frac{35}{60} - \frac{10}{60} \\ &= \frac{25}{60} \\ &= \frac{5}{12} \\ &= 0.4167. \end{aligned}$$

(f) The probability that the coach arrives at 23.00 depends on what is meant by "arrives at 23.00". If it has a strict meaning, that is, the coach arrives at exactly 23.00 (not even a billionth of a second out) then this event has probability zero. However, if this description means "to the nearest minute" then we need the probability that the coach arrives at 23.00 plus or minus half a minute. In terms of the random variable X, this probability is

$$\begin{aligned} P(-0.5 < X < 0.5) &= P(X < 0.5) - P(X \le -0.5) \\ &= \frac{0.5+15}{60} - \frac{-0.5+15}{60} \\ &= \frac{15.5}{60} - \frac{14.5}{60} \\ &= \frac{1}{60} \\ &= 0.0167. \end{aligned}$$

(g) The question states that the coach cannot arrive more than 45 minutes late and so the probability it arrives at 0.00 is zero.

(h) It does not seem very realistic to have sharp cutoffs at -15 and 45 minutes late. It would seem more realistic to let the probability density decrease gradually in both directions.

2. As the network server receives incoming requests according to a Poisson process with mean $\lambda = 2.5$ per minute, the time between successive requests X has an exponential distribution with parameter $\lambda = 2.5$ per minute.

   (a) The expected time between arrivals of requests is

$$E(X) = \frac{1}{\lambda} = \frac{1}{2.5} = 0.4 \text{ minutes.}$$

   (b) Probabilities for this distribution are calculated using

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-2.5 \times x} & \text{for } x > 0. \end{cases}$$

   The probability that the time between requests is less than 2 minutes is

$$P(X < 2) = 1 - e^{-2.5 \times 2} = 1 - e^{-5} = 1 - 0.0067 = 0.9933.$$

   (c) The probability that the time between requests is greater than 1 minute is

$$P(X > 1) = 1 - P(X < 1) = 1 - \left(1 - e^{-2.5 \times 1}\right) = e^{-2.5} = 0.0821.$$

   (d) The probability that the time between requests is between 30 seconds and 50 seconds is

$$\begin{aligned} P\left(\frac{1}{2} < X < \frac{5}{6}\right) &= P\left(X < \frac{5}{6}\right) - P\left(X \leq \frac{1}{2}\right) \\ &= 1 - e^{-2.5 \times 5/6} - \left(1 - e^{-2.5 \times 1/2}\right) \\ &= e^{-1.25} - e^{-12.5/6} \\ &= 0.2865 - 0.1245 \\ &= 0.1620. \end{aligned}$$

3. Let X denote the time to first breakdown. Then

   Company 1: X has an exponential distribution with $\lambda = 0.11$. Therefore, the probability of no breakdown within the first six months is

$$P(X > 6) = 1 - P(X < 6) = 1 - \left(1 - e^{-0.11 \times 6}\right) = e^{-0.66} = 0.5169.$$

Company 2: X has an exponential distribution with $\lambda = 0.01$. Therefore, the probability of no breakdown within the first six months is

$$P(X > 6) = 1 - P(X < 6) = 1 - \left(1 - e^{-0.01\times 6}\right) = e^{-0.06} = 0.9418.$$

On the basis of this calculation alone, recommend buy from Company 2 as their probability of no breakdown within the first six months is much larger than that of Company 1.

To take into account a difference in price you might consider the cost to the company of a breakdown within six months and compare the expected monetary values of the two possible decisions.