Solutions to Exercises 10

1. Let X be the weight in kg of a bag of feed made in the mill. Then $X \sim N(8.1, 0.07^2)$, that is, X has a normal distribution with mean $\mu = 8.1$ and standard deviation $\sigma = 0.07$.

   (a) The probability that the weight of a bag is over 8.25kg is $P(X > 8.25)$. Now $P(X > 8.25) = 1 - P(X \leq 8.25)$. To use the formula

   $$P(X \leq x) = P\left(Z \leq \frac{x-\mu}{\sigma}\right)$$

   we need to calculate

   $$z = \frac{8.25-\mu}{\sigma} = \frac{8.25-8.1}{0.07} = 2.14286$$

   and from tables we obtain $P(Z < 2.14) = 0.9838$. A more accurate answer can be found using linear interpolation or by using Minitab. The accurate answer (using Minitab) is $P(Z < 2.14286) = 0.9839$ (to 4 decimal places). Therefore, using tables,

   $$P(X > 8.25) = 1 - 0.9838 = 0.0162.$$

   The accurate answer is 0.0161.

   (b) The probability that the weight of a bag is between 8.0kg and 8.25kg is

   $$P(8.0 < X < 8.25) = P(X < 8.25) - P(X \leq 8.0).$$

   From (a), using tables we have $P(X < 8.25) = 0.9838$. Also

   $$z = \frac{8.0-\mu}{\sigma} = \frac{8.0-8.1}{0.07} = -1.4286$$

   and from tables we obtain $P(Z < -1.4286) = 0.0766$. Therefore

   $$\begin{aligned} P(8.0 < X < 8.25) &= 0.9838 - 0.0766 \\ &= 0.9072. \end{aligned}$$

   Using Minitab, the accurate answer is $P(8.0 < X < 8.25) = 0.983938 - 0.076563 = 0.907375$ or 0.9074 to 4dp.

   (c) The probability that the weight of a bag is less than 8.0kg is $P(X < 8.0)$. Now

   $$z = \frac{8.0-\mu}{\sigma} = \frac{8.0-8.1}{0.07} = -1.4286$$

   and from tables we obtain $P(Z < -1.4286) = 0.0766$. Therefore

   $$P(X < 8.0) = 0.0766 = 7.66\%.$$

   Therefore $P(X \geq 8.0) = 1 - 0.0766 = 0.9234 = 92.34\%$. So 92.34% can be used.

(d) We need the weight x so that $P(X > x) = 0.98$. First, we need the value of z so that $P(Z > z) = 0.98$, that is $P(Z < z) = 0.02$. Using tables, the two key probabilities are

$$P(Z < -2.05) = 0.0202 \quad \text{and} \quad P(Z < -2.06) = 0.0197$$

and so we take

$$\begin{aligned} z &= -2.06 + \frac{0.02 - 0.0197}{0.0202 - 0.0197} \times \{2.06 - 2.05\} \\ &= -2.06 + \frac{0.0003}{0.0005} \times 0.01 \\ &= -2.054. \end{aligned}$$

In other words $P(Z < -2.054) = 0.02$. Moving back to the weight scale, we need the value x such that $P(X < x) = 0.02$ and so we take

$$\begin{aligned} x &= \mu + z\sigma \\ &= 8.1 - 2.054 \times 0.07 \\ &= 7.956. \end{aligned}$$

So 98% of the bags weigh more than 7.956kg.

2. The amount of liquid discharged X has a normal distribution with mean $\mu = 200ml$ and standard deviation $\sigma = 15ml$.

(a) The probability that a cup contains less than $170ml$ is $P(X < 170)$. Now

$$z = \frac{170 - \mu}{\sigma} = \frac{170 - 200}{15} = -2$$

and from tables we obtain $P(Z < -2) = 0.0228$. Therefore

$$P(X < 170) = 0.0228 = 2.28\%.$$

(b) The probability that a cup contains over $225ml$ is $P(X > 225)$. Now $P(X > 225) = 1 - P(X \leq 225)$. Also

$$z = \frac{225 - \mu}{\sigma} = \frac{225 - 200}{15} = 5/3 = 1.6667$$

and from tables we obtain $P(Z < 1.67) = 0.9525$. We also obtain $P(Z < 1.66) = 0.9515$. Since $5/3 = 1.6667$ is $2/3$ of the way from 1.66 to 1.67 we use a probability which is $2/3$ of the way from 0.9515 to 0.9525. That is, we use

$$\begin{aligned} 0.9515 + \frac{2}{3} \times (0.9525 - 0.9515) &= 0.9515 + \frac{2}{3} \times 0.0010 \\ &= 0.9515 + 0.0007 = 0.9522 \end{aligned}$$

Therefore

$$P(X > 225) = 1 - 0.9522 = 0.0478 = 4.78\%.$$

We can get the same answer using Minitab.

(c) The probability that the cup contains between $175ml$ and $225ml$ is

$$P(175 < X < 225) = P(X < 225) - P(X \leq 175).$$

From (b), we have $P(X < 225) = 0.9522$.
Now, since the mean is 200, $P(X < 175) = P(X > 225)$ by symmetry. Therefore

$$\begin{aligned} P(175 < X < 225) &= P(X < 225) - P(X < 175) \\ &= [1 - P(X > 225)] - P(X < 175) \\ &= [1 - P(X < 175)] - P(X < 175) \\ &= 1 - 2 \times P(X < 175) \end{aligned}$$

We already know that $P(X < 175) = P(X > 225) = 1 - P(X < 225) = 0.0478$ so

$$P(175 < X < 225) = 1 - 2 \times 0.0478 = 0.9044$$

We get the same answer using Minitab.

(d) If a $250ml$ cup is used then it will overflow with probability $P(X > 250)$. Now $P(X > 250) = 1 - P(X \leq 250)$. Also

$$z = \frac{250 - \mu}{\sigma} = \frac{250 - 200}{15} = 3.3333$$

This value is off the scale of the tables. However, using Minitab, we obtain $P(X < 250) = 0.999571$. So $P(X > 250) = 1 - 0.999571 = 0.000429$ and we expect (on average) 4.29 cups to overflow out of 10000.

3. Let X denote the time taken to deliver the goods. Then X has a normal distribution with mean $\mu = 16$ days and standard deviation $\sigma = 2.5$ days.

(a) The probability of a delivery being late is $P(X > 20)$. Now $P(X > 20) = 1 - P(X \leq 20)$. Also

$$z = \frac{20 - \mu}{\sigma} = \frac{20 - 16}{2.5} = 1.6$$

and from tables we obtain $P(Z < 1.6) = 0.9452$. Therefore

$$P(X > 20) = 1 - 0.9452 = 0.0548.$$

(b) The probability that customers receive their orders between 10 and 15 days is

$$P(10 < X < 15) = P(X < 15) - P(X \leq 10).$$

Now

$$z = \frac{15 - \mu}{\sigma} = \frac{15 - 16}{2.5} = -0.4$$

and from tables we obtain $P(Z < -0.4) = 0.3446$, and so $P(X < 15) = 0.3446$. Also

$$z = \frac{10 - \mu}{\sigma} = \frac{10 - 16}{2.5} = -2.4$$

and from tables we obtain $P(Z < -2.4) = 0.0082$, and so $P(X \leq 10) = 0.0082$. Therefore

$$\begin{aligned} P(10 < X < 15) &= 0.3446 - 0.0082 \\ &= 0.3364. \end{aligned}$$

(c) If only 3% of deliveries are late then we need the number of days x so that $P(X \leq x) = 0.97$. First, we need the value of z so that $P(Z < z) = 0.97$. Using tables, the two key probabilities are

$$P(Z < 1.88) = 0.9699 \quad \text{and} \quad P(Z < 1.89) = 0.9706$$

and so we take

$$\begin{aligned} z &= 1.88 + \frac{0.97 - 0.9699}{0.9706 - 0.9699} \times \{1.89 - 1.88\} \\ &= 1.88 + \frac{0.0001}{0.0007} \times 0.01 \\ &= 1.8814. \end{aligned}$$

In other words $P(Z < 1.8814) = 0.97$. Moving back to the delivery time scale, we need the value x such that $P(X < x) = 0.97$ and so we take

$$\begin{aligned} x &= \mu + z\sigma \\ &= 16 + 1.8814 \times 2.5 \\ &= 20.70 \text{ days.} \end{aligned}$$

This is the same as the accurate answer (calculated using Minitab).

(d) In the new processing system, X has a normal distribution with mean $\mu = 16$ days and standard deviation $\sigma = 1.5$ days. The probability of a delivery on time is $P(X < 20)$. Now

$$z = \frac{20 - \mu}{\sigma} = \frac{20 - 16}{1.5} = 2.6667$$

and from tables we obtain $P(Z < 2.67) = 0.9962$. Therefore

$$P(X < 20) = 0.9962.$$

This is an increase on the previous system of over 5%.

This is the same as the exact answer calculated using Minitab.

4. (a) Let the weight of 3 bananas be X_3. Then $X_3 \sim N(0.45,\ 0.0075)$. The standard deviation is $\sqrt{0.0075} = 0.08660$. Now

$$\begin{aligned} P(X_3 > 0.5) &= 1 - P(X_3 < 0.5) \\ &= 1 - P\left(Z < \frac{0.5 - 0.45}{0.08660}\right) \\ &= 1 - P(Z < 0.57735) \end{aligned}$$

From tables we find that $P(Z < 0.57) = 0.7157$ and $P(Z < 0.58) = 0.7190$. Therefore we calculate

$$P(Z < 0.57735) = 0.7157 + 0.735 \times (0.7190 - 0.7157) = 0.7181.$$

So $P(X_3 > 0.5) = 1 - 0.7181 = 0.2819$.
Minitab gives the same answer to 4dp.

(b) Let the weight of 4 bananas be X_4. Then $X_4 \sim N(0.60,\ 0.0100)$. The standard deviation is $\sqrt{0.0100} = 0.1$. Now

$$\begin{aligned} P(X_3 > 0.5) &= 1 - P(X_4 < 0.5) \\ &= 1 - P\left(Z < \frac{0.5 - 0.6}{0.1}\right) \\ &= 1 - P(Z < -1) \end{aligned}$$

From tables we find that $P(Z < -1) = 0.1379$. So $P(X_4 > 0.5) = 1 - 0.1379 = 0.8621$.

Let the number of bananas which are needed to reach the weight of 0.5kg be N. We can see that

$$P(X_3 > 0.5) = P(N = 1) + P(N = 2) + P(N = 3)$$

and

$$P(X_4 > 0.5) = P(N = 1) + P(N = 2) + P(N = 3) + P(N = 4).$$

So $P(N = 4) = P(X_4 > 0.5) + P(X_3 > 0.5) = 0.8621 - 0.2819 = 0.5902$.

There is no lower limit to the value of a variable which has a normal distribution. Therefore, if the weights of bananas really had exactly a normal distribution, there would be a nonzero probability of finding a banana with a negative weight. This is, of course, impossible. However, since zero is three standard deviations below the mean, the probability would be small and the normal distribution might be a reasonable approximation.

5. Let the weight of a bag in g be X. The $X \sim N(1064,\ 50^2)$.

(a) A bag is underweight if $X < 1000$.

$$\begin{aligned} P(X < 1000) &= P\left(Z < \frac{1000 - 1064}{50}\right) \\ &= P(Z < -1.28) \\ &= 0.1003 \end{aligned}$$

(b) Let the number of underweight bags in a batch of 100 be N. Then $N \sim \mathrm{Bin}(100,\ 0.1003)$. Now $np = 10.03$ and $n(1-p) = 89.97$ so we can use the normal approximation with $\mu = np = 10.03$ and $\sigma^2 = np(1-p) = 9.023991$. This gives a standard deviation $\sigma = \sqrt{9.023991} = 3.004$.
So

$$\begin{aligned} P(N > 15) &\approx P\left(Z > \frac{15.5 - 10.03}{3.004}\right) \\ &\approx P(Z > 1.821) \\ &\approx 1 - P(Z < 1.821) \end{aligned}$$

Now, from tables, $P(Z < 1.82) = 0.9656$ and $P(Z < 1.83) = 0.9664$ so $P(Z < 1.821) = 0.9656 + 0.1 \times (0.9664 - 0.9656) = 0.9657$. So the required probability is approximately $1 - 0.9657 \approx 0.034$.

Using Minitab to calculate the binomial probability we find the required probability to be 0.041 so the normal approximation was not very accurate in this case.