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Contents

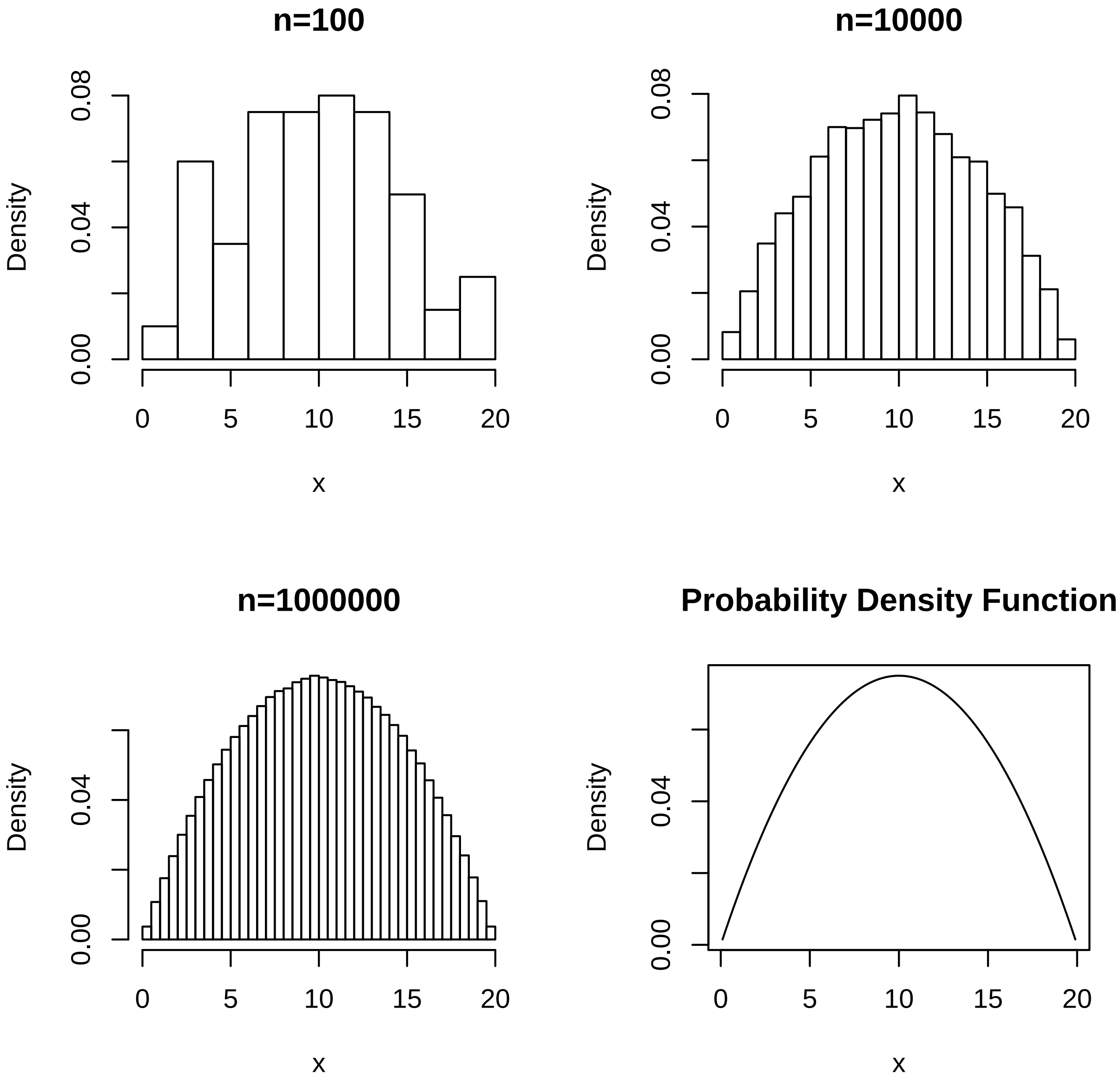
Chapter 9

Continuous Probability Models

9.1 Introduction

We have seen how discrete random variables can be modelled by discrete probability distributions such as the binomial and Poisson distributions. We now consider how to model continuous random variables. A variable is discrete if it takes a *countable* number of values, for example, $r = 0, 1, 2, \ldots, n$ or $r = 0, 1, 2, \ldots$ or $r = 0, 0.1, 0.2, \ldots, 0.9, 1.0$. In contrast, the values which a continuous variable can take form a continuous scale. One simple example of a continuous variable is height. Although in practice we might only record height to the nearest cm, if we could measure height exactly (to billions of decimal places) we would find that everyone had a different height. This is the essential difference between discrete and continuous variables. Therefore, if we could measure the exact height of every one of the n people on the planet, we would find that, for any height x, the proportion of people of height x is either $1/n$ or 0. And if we imagine the number of people on the planet growing over time ($n \to \infty$), this proportion tends to zero. This feature poses a problem for modelling continuous random variables as we can no longer use the methods we have seen work for discrete random variables.

The solution can be found by considering a (relative frequency) histogram of a sample of values taken by the continuous random variable, and thinking about what happens to the histogram as the sample size increases. For example, consider the following graphs which show histograms for samples of 100, 10000 and 1000000 observations made on a continuous random variable which can take values between 0 and 20. The final graph shows what happens when the sample size becomes infinitely big. This final graph is called the probability density function.



As the population sizes gets larger, the histogram intervals get smaller and the jagged profile of the histogram smooths out to become a curve. We call this curve the *probability density function (pdf)* and it is usually written as $f(x)$. Note that probabilities such as $P(X < x)$ can be determined using the pdf as they equate to areas under the curve.

The key features of pdfs are

1. pdfs never take negative values
2. the area under a pdf is one: $P(-\infty < X < \infty) = 1$
3. areas under the curve correspond to probabilities
4. $P(X \leq x) = P(X < x)$ since $P(X = x) = 0$.

We now consider some particular probability distributions that are often used to describe continuous random variables.

9.2 The Uniform Distribution

The *uniform distribution* is the most simple continuous distribution. As the name suggests, it describes a variable for which all possible outcomes are equally likely. For example, suppose you manage a group of Environmental Health Officers and need to decide at what time they should inspect a local hotel. You decide that any time during the working day (9.00 to 18.00) is okay but you want to decide the time "randomly". Here randomly is a short-hand for "a random time, where all times in the working day are equally likely to be chosen". Let X be the time to their arrival at the hotel measured in terms of minutes from the start of the day. Then X is a uniform random variable between 0 and 540:

The total area (base × height) under the pdf must equal one. Therefore, as the base is 540, the height must be $1/540$. Hence the probability density function (pdf) for the continuous random variable X is

$$f(x) = \begin{cases} \dfrac{1}{540} & \text{for } 0 \leq x \leq 540 \\ 0 & \text{otherwise.} \end{cases}$$

In general, we say that a random variable X which is equally likely to take any value between a and b has a uniform distribution on the interval a to b. The random variable has probability density function (pdf)

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and probabilities can be calculated using the formula

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b. \end{cases}$$

Therefore, for example, the probability that the inspectors visit the hotel in the morning (within 180 minutes after 9am) is

$$P(X \leq 180) = \frac{180-0}{540-0} = \frac{1}{3}.$$

The probability of a visit during the lunch hour (12.30 to 13.30) is

$$\begin{aligned} P(210 \leq X \leq 270) &= P(X \leq 270) - P(X < 210) \\ &= \frac{270-0}{540-0} - \frac{210-0}{540-0} \\ &= \frac{270-210}{540} \\ &= \frac{60}{540} \\ &= \frac{1}{9}. \end{aligned}$$

9.2.1 Mean and Variance

The mean and variance of a continuous random variable can be calculated in a similar manner to that used for a discrete random variable. However the specific techniques required to do this are outside the scope of this course and so we will simply state the results.

If X is a uniform random variable on the interval a to b then its mean and variance are

$$E(X) = \mu = \frac{a+b}{2}, \qquad Var(X) = \sigma^2 = \frac{(b-a)^2}{12}.$$

In the above example, we have

$$E(X) = \frac{a+b}{2} = \frac{0+540}{2} = 270,$$

so that the mean arrival of the inspectors is 9am $+ 270$ minutes $= 13.30$. Also

$$Var(X) = \frac{(540-0)^2}{12} = 24300$$

and therefore $SD(X) = \sqrt{Var(X)} = \sqrt{24300} = 155.9$ minutes.

9.3 The Exponential Distribution

The *exponential distribution* is another common distribution that is used to describe continuous random variables. It is often used to model lifetimes of products and times between "random" events such as arrivals of customers in a queueing system or arrivals of orders. The distribution has one parameter, λ. Its probability density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

and probabilities can be calculated using

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0. \end{cases}$$

The main features of this distribution are:

1. an exponentially distributed random variable can only take positive values
2. larger values are increasingly unlikely – exponential decay
3. the value of λ fixes the rate of decay – larger values correspond to more rapid decay.

Consider an example in which the time (in minutes) between successive users of a pay phone can be modelled by an exponential distribution with $\lambda = 0.3$. The probability of the gap between phone users being less than 5 minutes is

$$P(X < 5) = 1 - e^{-0.3\times 5} = 1 - 0.223 = 0.777.$$

Also the probability that the gap is more than 10 minutes is

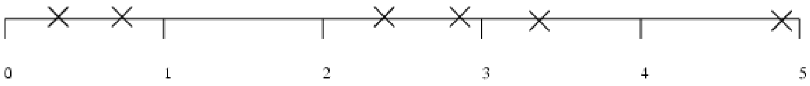
$$P(X > 10) = 1 - P(X \leq 10) = 1 - \left(1 - e^{-0.3\times 10}\right) = e^{-0.3\times 10} = 0.050$$

and the probability that the gap is between 5 and 10 minutes is

$$P(5 < X < 10) = P(X < 10) - P(X \leq 5) = 0.950 - 0.777 = 0.173.$$

One of the main uses of the exponential distribution is as a model for the times between events occurring randomly in time. We have previously considered events which occur at random points in time in connection with the Poisson distribution. The Poisson distribution describes probabilities for the number of events taking place in a given time period. The exponential distribution describes probabilities for the times between events. Both of these concern events occurring randomly in time (at a constant average rate, say λ). This is known as a *Poisson process*.

Consider a series of randomly occurring events such as calls at a credit card call centre. The times of calls might look like



There are two ways of viewing these data. One is as the number of calls in each minute (here 2, 0, 2, 1 and 1) and the other as the times between successive calls. For the Poisson process,

- the number of calls has a Poisson distribution with parameter λ, and
- the time between successive calls has an exponential distribution with parameter λ.

9.3.1 Mean and Variance

The mean and variance of the exponential distribution can be shown to be

$$E(X) = \mu = \frac{1}{\lambda}, \qquad Var(X) = \sigma^2 = \frac{1}{\lambda^2}.$$

9.4 Exercises 9

1. An express coach is due to arrive in Newcastle from London at 23.00. However in practice it is equally likely to arrive anywhere between 15 minutes early to 45 minutes late, depending on traffic conditions. Let the random variable X denote the amount of time (in minutes) that the coach is delayed.

   (a) Sketch the pdf.

   (b) Calculate the mean and standard deviation of the delay time.

   (c) What is the probability that the coach is less than 5 minutes late?

   (d) What is the probability that the coach is more than 20 minutes late?

   (e) What is the probability that the coach arrives between 22.55 and 23.20?

   (f) What is the probability that the coach arrives at 23.00?

   (g) What is the probability that the coach arrives at 0.00?

   (h) Do you think that this is a good model for the coach's arrival time?

2. A network server receives incoming requests according to a Poisson process with rate $\lambda = 2.5$ per minute.

   (a) What is the expectation of the time between arrivals of requests?

   (b) What is the probability that the time between requests is less than 2 minutes?

   (c) What is the probability that the time between requests is greater than 1 minute?

   (d) What is the probability that the time between requests is between 30 seconds and 50 seconds?

3. As Production Manager, you are responsible for buying a new piece of equipment for your company's production process. A salesman from one company has told you that he can supply you with equuipment for which the time to first breakdown (in months) follows an exponential distribution with $\lambda = 0.11$. Another salesman (from another company) has told you that the time to first breakdown of their machines is also exponentially distributed but with $\lambda = 0.01$. It is very important that the equipment you purchase does not break down for at least six months. Calculate the probability of this outcome for both suppliers and make a recommendation to the company board about which machine should be bought.

   How might you take into account a difference between the prices for the machines?