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Chapter 10

The Normal Distribution

10.1 Introduction

The *normal* distribution is possibly the best known and most used continuous probability distribution. It provides a good model for data in very many different applications, for example, the yields of crops, the heights of people, students' marks. The outcomes of many production processes also follow normal distributions and hence it is used widely in industry.

The normal distribution has two parameters: the mean μ and the variance σ^2 . The standard deviation $\sigma = \sqrt{\sigma^2}$ but we usually use the variance to specify the parameters. The probability density function of a normal distribution is often said to be “bell shaped” :

The formula for the pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}.$$

There is no simple formula for calculating probabilities. However, they can be determined using tables or statistical packages such as Minitab.

There are four important characteristics of the normal distribution:

1. It is symmetrical about its mean, μ .

2. The mean, median and mode all coincide.
3. The area under the curve is equal to 1.
4. The curve extends for ever in both directions to infinity (i.e. to $\pm\infty$).

Below is a plot of the pdf of normal distributions for different values of μ and σ .

Note that the mean μ locates the distribution on the x -axis and the standard deviation σ affects the spread of the distribution, with larger values giving flatter and wider curves.

10.2 Notation

If a random variable X has a normal distribution with mean μ and variance σ^2 , then we write

$$X \sim N(\mu, \sigma^2).$$

For example, a random variable X which follows a normal distribution with mean 10 and variance 25 is written as $X \sim N(10, 25)$ or $X \sim N(10, 5^2)$. It is important to note that the second parameter in this notation is the variance and not the standard deviation.

10.3 Some properties of normal variables

An important property concerns addition (or subtraction) of normal random variables. If X_1 and X_2 both have normal distributions and $Y = X_1 + X_2$ then Y also has a normal distribution. If $Z = X_1 - X_2$ then Z also has a normal distribution. If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ then Y has mean $\mu_y = \mu_1 + \mu_2$ and Z has mean $\mu_z = \mu_1 - \mu_2$. If X_1 and X_2 are *independent* then Y and Z both have variance $\sigma_y^2 = \sigma_z^2 = \sigma_1^2 + \sigma_2^2$.

For example, if wagonloads of material have weights which are normally distributed with mean 3 tonnes and standard deviation 0.5 tonnes then the total weight from two wagonloads is normally distributed with mean $3 + 3 = 6$ tonnes and standard deviation $\sqrt{0.5^2 + 0.5^2} = 0.7071$ tonnes.

If $X \sim N(\mu, \sigma^2)$ and a and b are fixed numbers and $W = a + bX$ then $W \sim N(a + b\mu, k^2\sigma^2)$. For example, if the weight of a load of material has a normal distribution with mean 3 tonnes and

standard deviation 0.5 tonnes and we take exactly 1 tonne out then the weight of the remainder has a normal distribution with mean 2 tonnes and standard deviation 0.5 tonnes. If the price we can expect to get for a certain quantity of our product is £ X and $X \sim N(1000, 50^2)$ and a pound is worth 1.6 euros then the price in euros is $Y \sim N(1600, 80^2)$.

An important consequence of this is that, if $X \sim N(\mu, \sigma^2)$, and

$$Z = \frac{X - \mu}{\sigma}$$

then $Z \sim N(0, 1)$.

10.3.1 Probability calculations and the standard normal distribution

All probabilities for the normal distribution can be expressed in terms of those for a normal distribution with mean 0 and variance 1. Usually, a random variable with this *standard normal distribution* is called Z , that is

$$Z \sim N(0, 1).$$

Probabilities for a random variable $X \sim N(\mu, \sigma^2)$ can be determined in terms of those for $Z \sim N(0, 1)$ using the formula

$$P(X \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

and values for $P(Z \leq z)$ can be found in tables (see the end of this chapter). Therefore, the probability $P(X \leq x)$ can be found by looking up (in standard normal tables) the probability corresponding to

$$z = \frac{x - \mu}{\sigma}.$$

We now look at a series of examples to illustrate how to calculate normal probabilities using these tables.

1. We first look at how to determine probabilities for the standard normal distribution.
 - (a) The probability that the random variable Z is less than -1.46 is $P(Z < -1.46)$. Therefore we look for the probability in tables corresponding to $z = -1.46$: row labelled -1.4 , column headed -0.06 . This gives $P(Z < -1.46) = 0.0721$.
 - (b) The probability that the random variable Z is less than -0.01 is $P(Z < -0.01)$. Therefore we look for the probability in tables corresponding to $z = -0.01$: row labelled 0.0 , column headed -0.01 . This gives $P(Z < -0.01) = 0.4960$.
 - (c) The probability that the random variable Z is less than 0.01 is $P(Z < 0.01)$. Therefore we look for the probability in tables corresponding to $z = 0.01$: row labelled 0.0 , column headed 0.01 . This gives $P(Z < 0.01) = 0.5040$.
 - (d) The probability that the random variable Z is greater than 1.5 is $P(Z > 1.5)$. Now $P(Z > 1.5) = 1 - P(Z \leq 1.5)$. Therefore we look for the probability in tables corresponding to $z = 1.5$: row labelled 1.5 , column headed 0.00 . This gives $P(Z < 1.5) = 0.9332$ and therefore $P(Z > 1.5) = 1 - P(Z < 1.5) = 1 - 0.9332 = 0.0668$.

(e) The probability that the random variable Z lies between than -1.2 and 1.5 is

$$\begin{aligned}P(-1.2 < Z < 1.5) &= P(Z < 1.5) - P(Z \leq -1.2) \\ &= 0.9332 - 0.1151 \\ &= 0.8181.\end{aligned}$$

We now consider how to determine probabilities for general normal distributions.

2. Suppose we are interested in the IQ of 18-20 year olds and that IQs follow a normal distribution with mean $\mu = 100$ and standard deviation $\sigma = 15$. Mathematically we let the random variable X denote the IQ of a randomly chosen person from this age group and then $X \sim N(100, 15^2)$. Probability statements about IQs can be made as follows.

(a) The probability that an 18-20 year old has an IQ less than 85 is $P(X < 85)$. Using the formula

$$P(X \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

we need to calculate

$$z = \frac{85 - \mu}{\sigma} = \frac{85 - 100}{15} = -1$$

and from tables we obtain $P(Z < -1) = 0.1587$. Therefore

$$P(X < 85) = 0.1587.$$

(b) The probability that an 18-20 year old has an IQ greater than 142 is $P(X > 142)$. Now $P(X > 142) = 1 - P(X \leq 142)$ and

$$z = \frac{142 - \mu}{\sigma} = \frac{142 - 100}{15} = 2.8.$$

Using tables, we see that $P(Z \leq 2.8) = 0.9974$ and so

$$P(X > 142) = 1 - 0.9974 = 0.0026.$$

3. Suppose that the vitamin C content per 100g tin of tomato juice is normally distributed with mean $\mu = 20\text{mg}$ and standard deviation $\sigma = 4\text{mg}$. Let X be the vitamin C content of a randomly chosen tin.

(a) The probability that the tin has less than 25mg of vitamin C is $P(X < 25)$. Now

$$z = \frac{25 - \mu}{\sigma} = \frac{25 - 20}{4} = 1.25$$

and from tables we obtain $P(Z < 1.25) = 0.8944$. Therefore

$$P(X < 25) = 0.8944.$$

(b) The probability that the tin has more than 25mg of vitamin C is $P(X > 25)$. Now $P(X > 25) = 1 - P(X \leq 25)$ and so

$$P(X > 25) = 1 - 0.8944 = 0.1056.$$

(c) The probability that the tin has between 18mg and 25mg of vitamin C is

$$Pr(18 < X < 25) = P(X < 25) - P(X \leq 18).$$

We can determine $P(X \leq 18)$ from tables using

$$z = \frac{18 - \mu}{\sigma} = \frac{18 - 20}{4} = -0.5,$$

giving $P(X \leq 18) = P(Z < -0.5) = 0.3085$. Therefore

$$Pr(18 < X < 25) = 0.8944 - 0.3085 = 0.5859.$$

We can also use the tables in reverse. For example, we might want to know below what value are 95% of the population. This is equivalent to determining the value of z that satisfies $P(Z < z) = 0.95$. From tables, we can see that $P(Z < 1.64) = 0.9495$ and $P(Z < 1.65) = 0.9505$. Therefore the value we want for z lies between 1.64 and 1.65. If a more accurate value is needed we can interpolate between these values: 0.95 is half-way between 0.9495 and 0.9505 and so we take $z = 1.645$. This is a more accurate answer and sufficient in most cases. However, the exact value for z can be found from more detailed tables or via a computer package such as Minitab. Here are some more examples.

1. Below what value does 10% of the standard normal population fall? From tables we get

$$P(Z < -1.28) = 0.1003 \quad \text{and} \quad P(Z < -1.29) = 0.0985$$

and so we take

$$\begin{aligned} z &= -1.29 + \frac{0.1 - 0.0985}{0.1003 - 0.0985} \times \{-1.28 - (-1.29)\} \\ &= -1.29 + \frac{0.0015}{0.0018} \times 0.01 \\ &= -1.2817. \end{aligned}$$

In other words $P(Z < -1.2817) = 0.1$ and so 10% of the standard normal population falls below -1.2817 .

2. A similar calculation can be used to calculate the IQ that identifies the 10% of 18-20 year olds with the smallest IQ. We need the value of x , where $P(X < x) = 0.1$. Now this population has $\mu = 100$ and $\sigma = 15$. Also

$$P(X \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

and so we need x so that

$$P\left(Z \leq \frac{x - 100}{15}\right) = 0.1.$$

We know (from earlier) that $P(Z < -1.2817) = 0.1$ and therefore we solve

$$\frac{x - 100}{15} = -1.2817,$$

that is

$$\begin{aligned}x &= 100 - 1.2817 \times 15 \\ &= 100 - 19.2255 \\ &= 80.7745.\end{aligned}$$

Notice that the calculation that transforms the z -value onto the x -scale is

$$x = \mu + z\sigma.$$

3. What is the IQ that identifies the 1% of 18-20 year olds with the greatest IQ? Again, we first determine the value z that identifies the top 1% of a standard normal population and then translate this into an IQ. So we need the value z that satisfies $P(Z > z) = 0.01$. This is the same value as satisfies $P(Z < z) = 0.99$. A quick examination of tables gives the two key probabilities as

$$P(Z < 2.32) = 0.9898 \quad \text{and} \quad P(Z < 2.33) = 0.9901$$

and so we take

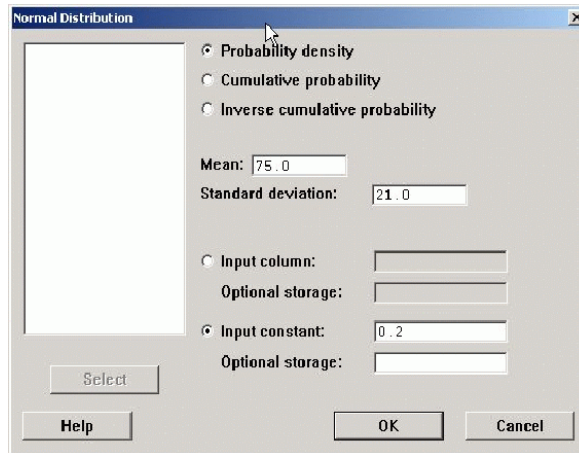
$$\begin{aligned}z &= 2.32 + \frac{0.99 - 0.9898}{0.9901 - 0.9898} \times \{2.33 - 2.32\} \\ &= 2.32 + \frac{0.0002}{0.0003} \times 0.01 \\ &= 2.3267.\end{aligned}$$

In other words $P(Z < 2.3267) = 0.99$ and so 1% of the standard normal population lies above $z = 2.3267$. Moving back to the IQ scale, we need the value x such that $P(X > x) = 0.01$ and so we take

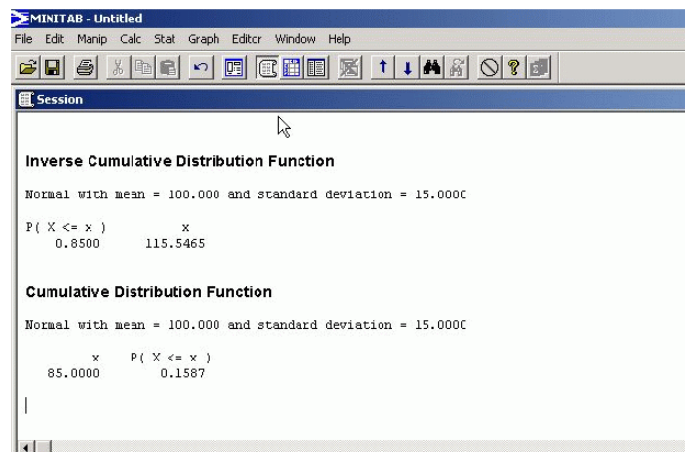
$$\begin{aligned}x &= \mu + z\sigma \\ &= 100 + 2.3267 \times 15 \\ &= 134.9.\end{aligned}$$

Minitab is very helpful with calculating normal probabilities. The following commands will calculate probabilities $P(X < x)$ and also values of x that satisfy $P(X < x) = p$:

1. Calc > Probability Distributions > Normal
opens up dialogue box



2. Select Cumulative probability for $P(X < x)$ or Inverse cumulative probability for the value of x satisfying $P(X < x) = p$
3. Enter the Mean (μ) and the Standard Deviation (σ)
4. Select Input Constant and enter the value for x or p (as appropriate)
5. Click OK
6. The answer is displayed in the Session Window:



10.4 The normal approximation to the binomial distribution

The normal distribution can be used as an approximation to the binomial distribution $\text{Bin}(n, p)$ for large n and medium p . (Say if both np and $n(1 - p)$ are greater than about 7). To approximate

a $\text{Bin}(n, p)$ distribution we use a normal distribution with the same mean and variance. That is $\mu = np$ and $\sigma^2 = np(1 - p)$.

Example. We plan to do a market research survey in which people will be asked whether or not they would buy a new product. A random sample of 600 people will be asked. Suppose that the true proportion of people in the population who would buy the product is 40%, i.e. $p = 0.4$. Find the probability that, in our survey, between 220 and 260 (inclusive) answer “Yes.”

In this case $n = 600$ and $p = 0.4$ so $np = 240$ and $n(1 - p) = 360$. Therefore we can use the approximation. We use the normal distribution with mean $\mu = np = 240$ and variance $\sigma^2 = np(1 - p) = 144$. That is $N(240, 144)$. The standard deviation is $\sigma = \sqrt{144} = 12$.

If $X \sim N(240, 144)$ then

$$\begin{aligned} \Pr(X < 220) &= \Pr\left(\frac{X - 240}{12} < \frac{220 - 240}{12}\right) \\ &= \Pr(Z < -1.67) \\ &= 0.0475. \end{aligned}$$

So $\Pr(\text{fewer than 220 “Yes”}) \approx 0.0475$.

We can sometimes obtain a better approximation by using a *continuity correction*. Since $B(n, p)$ is a discrete distribution, $\Pr(\text{number of “Yes”} = 220) > 0$. Using the continuity correction, we count as “220” everything between 219.5 and 220.5 so, for $\Pr(\text{number of “Yes”} < 220)$ we would use

$$\Pr(X < 219.5) = \Pr\left(Z < \frac{219.5 - 240}{12}\right) = \Pr(Z < -1.71) \approx 0.0436.$$

Here the continuity correction makes a noticeable difference. Sometimes it does not.

In the same way

$$\Pr(X < 260.5) = \Pr\left(Z < \frac{260.5 - 240}{12}\right) = \Pr(Z < 1.71) \approx 0.9564 = 1 - 0.0436.$$

So the probability that, in our survey, between 220 and 260 (inclusive) people say “Yes” is approximately $0.9564 - 0.0436 = 0.9128 \approx 0.91$.

10.5 Exercises 10

1. The weights of bags of animal feed made in a mill follow a normal distribution with mean $\mu = 8.1$ kg and standard deviation $\sigma = 0.07$ kg.
 - (a) What is the probability that the weight of a bag is over 8.25kg?
 - (b) What is the probability that the weight of a bag is between 8.0kg and 8.25kg?
 - (c) A customer requires bags which weigh no less than 8.0kg. What percentage of the mill's output can be used to supply this customer?
 - (d) The mill is trying to negotiate a new contract with this customer. It is in the mill's interests to be able to supply 98% of its output to the customer. What is the largest weight which achieves this requirement?
2. A drinks machine is regulated by its manufacturer so that it discharges an average of 200ml per cup. However, the machine is not particularly accurate and actually discharges an amount that has a normal distribution with standard deviation 15ml.
 - (a) What percentage of cups contain below the minimum permissible volume of 170ml?
 - (b) What percentage of cups contain over 225ml?
 - (c) What is the probability that the cup contains between 175ml and 225ml?
 - (d) How many cups would you expect to overflow if 250ml cups are used for the next 10000 drinks?
3. A company promises delivery within 20 working days of receipt of order. However in reality they deliver according to a normal distribution with a mean of 16 days and a standard deviation of 2.5 days.
 - (a) What proportion of customers receive their order late?
 - (b) What proportion of customers receive their orders between 10 and 15 days of placing their order?
 - (c) How many days should the delivery promise be adjusted to if only 3% of orders are to be late?
 - (d) A new order processing system promises to reduce the standard deviation of delivery times to 1.5 days. If this system is used, what proportion of customers will receive their deliveries within 20 days?
4. Bananas of a certain variety have weights, in kg, which are independent and normally distributed with mean 0.15 and variance 0.0025. Find the probability that
 - (a) three bananas weigh more than 0.5kg.
 - (b) four bananas weigh more than 0.5kg.

In a shop bananas are put on the scales one at a time until the total weight becomes more than 0.5kg. Find the probability that exactly four bananas are needed.

Explain why the first sentence in this question can only be approximately, not exactly, true.

5. Bags of produce have a nominal weight of 1kg. In fact the weights have a normal distribution with mean 1064g and standard deviation 50g. A bag which weighs less than 1kg is considered to be underweight.
- (a) Find the probability that a bag is underweight.
 - (b) Assuming that the weights of bags are independent, find an approximate value for the probability that, in a batch of 100 bags, more than 15 are underweight.

Probability Tables for the Standard Normal Distribution

The table contains values of $Pr(Z < z)$, where $Z \sim N(0, 1)$.

z	-0.09	-0.08	-0.07	-0.06	-0.05	-0.04	-0.03	-0.02	-0.01	0.00
-2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
-2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
-2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
-2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
-2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
-2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
-2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
-2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
-2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
-2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
-1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
-1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
-1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
-1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
-1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
-1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
-1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
-1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
-1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
-1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
-0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
-0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
-0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
-0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
-0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
-0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
-0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
-0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
-0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986