

10 Normal Distribution Means: Student's t Distribution

10.1 Student's t distribution

10.1.1 Introduction

In this section we will look at tests and confidence intervals for means of normal distributions. It is usual in practice for both the means and variances to be unknown. In this case the tests and confidence intervals are based on "Student's t Distribution."

10.1.2 Example 1

Fifty-four PM10 measurements (a measure of pollution from motor vehicle exhausts) are made on the vehicle deck of a ferry. It has been claimed that the mean PM10 value at this location is 10. The values observed were as follows.

14	11	13	13	13	15	11	16	10
13	14	11	13	12	10	14	10	14
16	14	14	11	11	11	13	12	13
11	11	15	14	16	12	17	9	16
11	19	14	12	12	10	11	12	13
13	14	11	11	15	12	16	15	11

Is there any evidence that the mean PM10 value is greater than 10?

10.2 One-sample t -test

10.2.1 Introduction

Let us suppose that we can make the following assumptions about the data in the example:

- They are **independent** observations.
- They are taken from a **normal** distribution.

Suppose we call the mean of this normal distribution μ . We can then test the null hypothesis that $\mu = 10$ against the alternative hypothesis $\mu > 10$. (This is a one-sided alternative. The two-sided alternative would be $\mu \neq 10$). We call this a one-sample test because we are comparing the sample mean, \bar{y} , from one sample with a theoretical population mean, $\mu_0 = 10$.

To find the significance of the difference we must standardise it by dividing by the standard deviation (called the *standard error*) of \bar{y} . The actual standard error of \bar{y} is $\sqrt{\sigma^2/n}$ (where n is the number of observations in the sample, in this case 54, and σ^2 is the variance of the distribution from which the observations were taken) but the value of σ^2 is unknown so we use instead an estimate $\sqrt{s^2/n}$. Because of this the distribution of our test statistic, t , is not normal but *Student's t on $n - 1$ degrees of freedom* or t_{n-1} for short. We compare our value for t with figures given in a t -table for the appropriate number of degrees of freedom.

10.2.2 Example 1 continued

We calculate

$$\begin{aligned}\sum y &= 695 \\ \sum y^2 &= 9177 \\ \bar{y} &= 12.87 \\ s^2 &= 4.3791\end{aligned}$$

To test the null hypothesis $H_0 : \mu = 10$ against the alternative $H_A : \mu > 10$ we calculate the test statistic

$$t_{53} = \frac{12.87 - 10}{\sqrt{4.3791/54}} = 10.08.$$

The upper 0.1 % point of the t_{53} distribution is 3.252 so the result is significant at the 0.1 % level. We therefore reject H_0 and conclude that $\mu > 10$.

We can calculate a 95 % lower confidence bound for μ as

$$\mu > \bar{y} - 1.674\sqrt{4.3791/54}.$$

That is $\mu > 12.7$.

Note that for a two-sided test the 0.1 % critical value is 3.485 so the result would still be significant at the 0.1 % level. A two-sided 95 % confidence interval would be $\bar{y} \pm 2.006\sqrt{s^2/54}$. That is $12.3 < \mu < 13.4$.

10.3 Notes

1. The *central limit theorem* tells us that as the sample size increases the distribution of the mean of a sample from a distribution, which need not be normal itself (but subject to certain conditions), tends to a normal distribution. In fact, for many distributions, convergence to normality is quite rapid. Therefore t tests are often used even if the distribution from which the observations are taken is not exactly normal and even if the sample is not very large. This is not necessarily always a good idea though. Other tests are available which do not require the assumption of a normal distribution. If no particular distribution is assumed the tests are called *distribution-free* or *nonparametric*.
2. As ν , the number of degrees of freedom, increases, the t_ν distribution tends to a standard normal distribution, so when ν is large the critical points of t_ν are approximately the same as those of $N(0, 1)$.

10.4 The Two-sample t Test

10.4.1 Introduction

We have already looked at making inferences about the mean of a normal distribution. Now we turn our attention to the difference between the means of two normal distributions. For example, in an experiment to examine whether the breaking strengths of ropes supplied by two manufacturers are different, we might wish to compare the mean strenghts in samples from the two.

Suppose we have two samples of observations. The n_1 observations, $Y_{11}, \dots, Y_{n_1 1}$, in the first sample are independent and identically distributed from a $N(\mu_1, \sigma^2)$ distribution. The n_2 observations, $Y_{12}, \dots, Y_{n_2 2}$, in the second sample are independent and identically distributed from a $N(\mu_2, \sigma^2)$ distribution. The observations in the two sample are independent of each other. The three parameters, μ_1, μ_2 and σ^2 , are all unknown. Notice that we are assuming:

- Independence,
- Normality,
- The two distributions have the same variance.

We can use the t distribution to test, for example, the hypothesis that $\mu_1 = \mu_2$.

There is also a test to use when we can not assume that the variances are equal.

We can test the null hypothesis $H_0 : \mu_1 = \mu_2$ by comparing $\bar{y}_1 - \bar{y}_2$ with zero. Here \bar{y}_1 and \bar{y}_2 are the sample means for the two samples. Provided that we can assume that the two population variances are equal, the test statistic has a t -distribution on $n_1 + n_2 - 2$ degrees of freedom, where n_1 and n_2 are the two sample sizes. If we do not assume that the two population variances are equal then the number of degrees of freedom is reduced.

Similarly we can construct confidence intervals.

10.4.2 Example 1

Suppose a change is made to the ventilation system on the ferry in Example 1 (Section 10.1.2). Then a further sample of 64 PM10 measurements is taken. The data are as follows.

13	9	13	9	9	13	9	9
10	8	13	7	7	9	9	13
14	7	10	8	10	8	8	9
12	11	8	10	11	12	12	7
11	9	8	11	12	8	9	10
8	6	11	8	6	11	14	9
10	9	6	9	9	11	9	11
11	7	12	8	10	9	9	8

Is there any evidence of a change in the mean PM10 value?
For the new data

$$\begin{aligned}\sum y &= 616 \\ \sum y^2 &= 6180 \\ \bar{y}_2 &= 9.625\end{aligned}$$

The value of $\sum(y - \bar{y})^2$ for this new sample is thus

$$6180 - (616)^2/64 = 251.$$

The corresponding value for Sample 1 is 232.0926. We calculate the *pooled* variance estimate

$$s^2 = \frac{232.0926 + 251.0000}{116} = 4.1646.$$

Our test statistic for the test against the two-sided alternative $H_A : \mu_1 \neq \mu_2$ is

$$t_{116} = \left| \frac{12.87 - 9.625}{\sqrt{4.1646(1/54 + 1/64)}} \right| = 8.605.$$

For a 0.1 % two-sided test the critical value of t_{116} is 3.379 so the result is very highly significant. We reject H_0 . There is strong evidence of a difference. The mean value is lower after the change.

A 95 % confidence interval for $\mu_1 - \mu_2$ is given by $12.87 - 9.625 \pm 1.981\sqrt{s^2(1/54 + 1/64)}$. That is $2.50 < \mu_1 - \mu_2 < 3.99$.

(We can also have one-sided tests and confidence intervals).

10.5 Problems

1. A boat builder would like to use Norwegian pine for part of a boat. He arranges for measurements of the failure stress of twenty standard samples. The data, in Nmm^{-2} , are as follows.

17.2	39.3	21.1	29.5	28.8	21.7	20.7	39.2	30.7	27.0
19.4	26.8	29.6	28.0	20.3	32.4	19.1	28.6	18.0	25.1

Assuming that the observations are independent and normally distributed, test the null hypothesis that the mean failure stress is 23 Nmm^{-2} against the alternative that it is greater than this and give a 95% one-sided confidence interval, of the form $\mu > x$, for the true mean.

2. Measurements are made of the hull surface roughness of the wetted sides two vessels treated with different paints. Each measurement consists of the height (microns) from the lowest trough to the highest peak in a length of 50mm. The data are as follows.

Paint A									
89	87	105	110	95	83	78	102	96	89
99	94	88	85	91	101	97	90	116	71
Paint B									
86	92	82	100	78	80	93	85	83	97
96	77	84	76	82	88	80	72	74	89

Assuming that the observations are independent and normally distributed and that the two population variances are equal, test the hypothesis that the population mean roughnesses are equal against the alternative that they are not and give a symmetric 95% confidence interval for the difference in the population means.

Do you think that the assumptions are likely to be valid in this case?