Excision in the cohomology of Banach algebras with coefficients in dual bimodules

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Abstract. We prove that, for every extension of Banach algebras \( 0 \to I \to A \to A/I \to 0 \) such that \( I \) has a bounded approximate identity and for every Banach \( A \)-bimodule \( X \) such that \( IX = XI \), there exist associated long exact sequences of continuous homology groups

\[
\cdots \to H_n(I, IX) \to H_n(A, X) \to H_n(A/I, X/IX) \to H_{n-1}(I, IX) \to \cdots
\]

and continuous cohomology groups

\[
\cdots \to H^{n-1}(I, (IX)^*) \to H^n(A/I, (X/IX)^*) \to H^n(A, X^*) \to H^n(I, (IX)^*) \to \cdots
\]

Note that, by the extension of the Cohen factorization theorem, \( IX = \{ b \cdot x : b \in I, x \in X \} \). In particular, the extension has the excision property in simplicial (co)homology. It also follows that the \( n \)-amenability of \( A \) implies the \( n \)-amenability of \( A/I \) and \( I \).


1. Introduction

In recent years the excision property in different kinds of cyclic and simplicial (co)homology was studied in \([Wo2, Wo3, CQ, Cu, Br1, Br2, Ly2]\). We say that an extension of Banach algebras has the excision property in a particular (co)homology if there is an associated long exact sequence of the corresponding (co)homology groups. The excision property gives a promising approach to the calculation of (co)homology groups. We can break the calculation down by making use of extensions of Banach algebras.

Let \( 0 \to I \to A \to A/I \to 0 \) be an extension of Banach algebras such that \( I \) has a bounded approximate identity and let \( X \) be a Banach \( A \)-bimodule \( X \) such that \( IX = XI \). Since \( I \) has a bounded approximate identity, the set \( \{ b \cdot x : b \in I, x \in X \} \) is precisely the closure of linear subspace \( IX \) of \( X \) \([Hew, CF-T]\), see also \([Jo, Proposition 1.8]\) and \([Pa, Theorem 5.2.2]\). Here we establish (Theorem 3.1)
that this extension has the excision property in continuous cohomology groups $H^n(A, X^*)$ where $X^*$ is a dual Banach $A$-bimodule, that is, that there is a long exact sequence of continuous cohomology groups

$$\cdots \rightarrow \mathcal{H}^{n-1}(I, (TX)^*) \rightarrow \mathcal{H}^n(A/I, (X/IX)^*) \rightarrow \mathcal{H}^n(A, X^*) \rightarrow \mathcal{H}^n(I, (TX)^*) \rightarrow \cdots$$

We also show (Theorem 4.4) that, under the same conditions on an extension and a bimodule $X$, the existence of an associated long exact sequence of continuous cohomology groups $H^n(A, X^*)$ is equivalent to the existence of one for continuous homology groups $H^n(A, X)$. From these two results we deduce the excision property in continuous homology groups $H^n(A, X^*)$, that is, we prove the existence of a long exact sequence of continuous homology groups

$$\cdots \rightarrow H_n(I, TX) \rightarrow H_n(A, X) \rightarrow H_n(A/I, X/IX) \rightarrow H_{n-1}(I, TX) \rightarrow \cdots$$

There is a broad literature on the amenability of Banach algebras [Jo, Ha], see [Pat1] for many references. In recent years this subject has taken several different directions: $n$-amenability [Pat2, PaSm, Da]; weakly amenability [BCD, Gr]; and simplicial triviality of Banach algebras [ChSi, Wo1, Ly1, Ly2]. The excision property, which we establish, is effective for the study of the above-mentioned properties of a Banach algebra. In particular, it follows that an extension of Banach algebras $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$, such that $I$ has a bounded approximate identity, has the excision property in simplicial cohomology $H^n(A, A^*)$ and homology $H_n(A, A)$ groups (cp. [Ly2]). Recall that the simplicial (co)homology are very useful for the calculation of the cyclic (co)homology [Co1, Co2, He2]. One can see that there is also a close connection between simplicial triviality of the Banach algebras $A$, $A/I$ and $I$. We say that $A$ is $n$-amenable if $H^n(A, X^*) = \{0\}$ for every Banach $A$-bimodule $X$ (see [Pat2, PaSm]). We prove (Corollary 5.3) that the $n$-amenability of $A$ implies the $n$-amenability of the two Banach algebras $A/I$ and $I$ when $I$ has a bounded approximate identity.

2. Definitions and notation

We recall some notation and terminology used in homological theory. These can be found in any text book on homological algebra, for instance MacLane [Ma], Loday [Lo] and Helemskii [He1] for the continuous case.

A chain complex $\mathcal{K}_\sim$ in the category of [Banach] linear spaces is a sequence of [Banach] linear spaces and [continuous] linear operators

$$\cdots \leftarrow K_n \xrightarrow{d_n} K_{n+1} \xleftarrow{d_{n+1}} K_{n+2} \xrightarrow{d_{n+2}} \cdots$$

such that $d_n \circ d_{n+1} = 0$ for every $n$. The cycles are the elements of $Z_n = \text{Ker} (d_{n+1} : K_n \rightarrow K_{n+1})$. The boundaries are the elements of $B_n = \text{Im} (d_n : K_{n+1} \rightarrow K_n)$. The relation $d_{n-1} \circ d_n = 0$ implies $B_n \subseteq Z_n$. The homology groups are defined by $H_n(\mathcal{K}_\sim) = Z_n/B_n$. 

A continuous morphism of chain complexes $\psi_n : K_n \to \mathcal{P}_n$ in the category of [Banach] linear spaces is a collection of [continuous] linear operators $\psi_n : K_n \to \mathcal{P}_n$ such that the following diagram is commutative for any $n$

$$
\begin{array}{ccc}
K_n & \xrightarrow{d_{n-1}} & K_{n-1} \\
\downarrow \psi_n & & \downarrow \psi_{n-1} \\
\mathcal{P}_n & \xrightarrow{d_{n-1}} & \mathcal{P}_{n-1}.
\end{array}
$$

Such a morphism obviously induces a map $H_n(\psi_n) : H_n(K_n) \to H_n(\mathcal{P}_n)$. A [continuous] morphism of chain complexes $\phi_n : K_n \to \mathcal{P}_n$ of [Banach] linear spaces is a [topological] quasi-isomorphism if $H_n(\psi_n) : H_n(K_n) \to H_n(\mathcal{P}_n)$ is a [topological] isomorphism for every $n$.

We shall use the similar terminology in the case of cochain complexes. Throughout the paper $id$ denotes the identity operator. For any a Banach algebra $A$ not necessarily unital, $A_+$ is the Banach algebra obtained by adjoining an identity to $A$. An extension of Banach algebras is a short exact sequence $0 \to B \to A \to D \to 0$ of Banach algebras and continuous homomorphisms.

Let $X$ be a Banach $A$-bimodule and let $S$ be a subset of $A$. Then $S^2$ is the linear span of the set $\{a_1 \cdot a_2 : a_1, a_2 \in S\}$, $SX$ is the linear span of the set $\{a \cdot x : a \in S, x \in X\}$ and $\overline{S}X$ is the closure in $X$ of the linear span of the set $\{a \cdot x : a \in S, x \in X\}$. Expressions of the type $\overline{X}S$ and $\overline{SX}$ have similar meanings.

We denote the projective tensor product of Banach spaces by $\hat{\otimes}$ (see, for example, [Sch], [He1, II.4.1]) and the projective tensor product of left and right Banach $A$-modules by $\hat{\otimes}_A$ (see [Rie]). Note that by $Z^{\otimes 0} \hat{\otimes} Y$ we mean $Y$, by $Z^{\otimes 1}$ we mean $Z$ and by $A^{\hat{\otimes} n}$ we mean the $n$ fold projective tensor power $A \hat{\otimes} \ldots \hat{\otimes} A$ of $A$. The algebra $A^e = A_+ \hat{\otimes}_A A_+^{op}$ is called the enveloping algebra of $A$; $A_+^{op}$ is the opposite algebra of $A_+$ with multiplication $a \cdot b = ba$. Every Banach $A$-bimodule can be regarded as a unital left Banach $A^e$-module. In the case when the Banach algebra $A$ is unital, the enveloping algebra is $A_+^{eun} = A \hat{\otimes} A^{op}$.

The category of Banach spaces is denoted by $Ban$; the category of left Banach $A$-modules is denoted by $A$-mod; the category of Banach $A$-bimodules is denoted by $A$-mod-$A$. Let $\mathcal{K}$ be one of the above categories of Banach $A$-modules and their morphisms. For $X,Y \in \mathcal{K}$, the Banach space of morphisms from $X$ to $Y$ is denoted by $\text{Hom}_\mathcal{K}(X,Y)$.

A complex of Banach $A$-modules and their morphisms is called admissible if it splits as a complex of Banach spaces [He1, III.3.11]. A complex of Banach $A$-modules and their morphisms is called weakly admissible if the dual complex splits as a complex of Banach spaces (see [He1, VII.1.16]). Thus a short exact sequence of Banach spaces and continuous linear operators $0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0$ is admissible if there exists a continuous operator $\alpha : W \to Z$ such that $j \circ \alpha = \text{id}_W$. Recall that admissibility is equivalent to the existence of continuous operators $\beta : Z \to Y$ and $\alpha : W \to Z$ such that $\beta \circ i = \text{id}_Y$, $j \circ \alpha = \text{id}_W$ and $i \circ \beta + \alpha \circ j = \text{id}_Z$. A short exact sequence of Banach spaces and continuous linear operators $0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0$
is weakly admissible if the dual short sequence \( 0 \to W^* \xrightarrow{\iota^*} Z^* \xrightarrow{\iota} Y^* \to 0 \) is admissible.

A module \( P \in \mathcal{K} \) is called projective in \( \mathcal{K} \) if, for each module \( Y \in \mathcal{K} \) and each epimorphism of modules \( \varphi \in h_{\mathcal{K}}(Y, P) \) such that \( \varphi \) has a right inverse as a morphism of Banach modules, there exists a right inverse morphism of Banach \( A \)-modules from \( \mathcal{K} \). For \( X \in \mathcal{K} \) a complex

\[
0 \leftarrow X \xrightarrow{\varepsilon} P_0 \xrightarrow{\phi_0} P_1 \xleftarrow{\phi_1} P_2 \xleftarrow{\phi_2} \cdots \quad (0 \leftarrow X \leftarrow \mathcal{P})
\]

over \( X \) is called a projective resolution in \( \mathcal{K} \) if it is admissible and all the modules in \( \mathcal{P} \) are projective in \( \mathcal{K} \). We shall denote the \( n \)th cohomology of the complex \( h_{\mathcal{K}}(\mathcal{P}, Y) \) where \( Y \in \mathcal{K} \) by \( \text{Ext}^n_{\mathcal{K}}(X, Y) \).

A module \( Y \in A\text{-mod} \) is called flat if for any admissible complex \( \mathcal{X} \) of right Banach \( A \)-modules the complex \( \mathcal{X} \otimes_A Y \) is exact. A module \( Y \in A\text{-mod} \) is called flat if for any admissible complex \( \mathcal{X} \) of Banach \( A \)-bimodules the complex \( \mathcal{X} \otimes_A Y \) is exact. For \( X \in \mathcal{K} \) a complex

\[
0 \leftarrow X \xrightarrow{\varepsilon} Q_0 \xrightarrow{\phi_0} Q_1 \xleftarrow{\phi_1} Q_2 \xleftarrow{\phi_2} \cdots \quad (0 \leftarrow X \leftarrow \mathcal{Q})
\]

over \( X \) is called a pseudo-resolution in \( \mathcal{K} \) if it is weakly admissible and all the modules in \( \mathcal{Q} \) are flat in \( \mathcal{K} \).

Let \( A \) be a Banach algebra and let \( X \) be a Banach \( A \)-bimodule. We will define the continuous homology \( \mathcal{H}_n(A, X) \) of the algebra \( A \) with coefficients in \( X \) (see, for example, [Jo], [He1, II.5.28]). We denote by \( C_n(A, X) \), \( n = 0, 1, \ldots \), the Banach space \( \mathcal{X} \otimes_A X \); we shall call the elements of this space \( n \)-dimensional continuous homology groups. We also set \( C_0(A, X) = X \). From the chains we form the standard homology complex

\[
0 \leftarrow C_0(A, X) \xrightarrow{d_0} \cdots \xrightarrow{d_n} C_n(A, X) \xleftarrow{d_{n+1}} \cdots \quad (\mathcal{C}_n(A, X))
\]

where the differential \( d_n \) is given by the formula

\[
d_n(x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) = (x \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_{n+1}) + \sum_{i=1}^{n} (-1)^i (x \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) + (-1)^{n+1} (a_{n+1} \cdot x \otimes a_1 \otimes \cdots \otimes a_n).
\]

The \( n \)-dimensional homology of \( \mathcal{C}_n(A, X) \), denoted by \( \mathcal{H}_n(A, X) \), is called the \( n \)-dimensional continuous homology group of the Banach algebra \( A \) with coefficients in \( X \).

For a Banach algebra \( A \) and a Banach \( A \)-bimodule \( X \), we define an \( n \)-cochain to be a bounded \( n \)-linear operator of \( A \times \cdots \times A \) into \( X \) and we denote the space of \( n \)-cochains by \( C^n(A, X) \). For \( n = 0 \) the space \( C^0(A, X) \) is defined to be \( X \). Let us consider the standard cohomological complex

\[
0 \to C^0(A, X) \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^n} C^n(A, X) \xrightarrow{\delta^n} C^{n+1}(A, X) \to \cdots \quad (\mathcal{C}^\ast(A, X))
\]

where the coboundary operator \( \delta^n \) is defined by

\[(\delta^n f)(a_1, \ldots, a_{n+1}) = a_1 \cdot f(a_2, \ldots, a_{n+1}) + \]
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\[
\sum_{i=1}^{n} (-1)^i f(a_1, ..., a_i a_{i+1}, ..., a_{n+1}) + (-1)^{n+1} f(a_1, ..., a_n) \cdot a_{n+1}.
\]

The \( n \)th cohomology group of \( C^\sim(A, X) \), denoted by \( H^n(A, X) \), is called the \( n \)-dimensional continuous cohomology group of \( A \) with coefficients in \( X \) (see, for example, [Jo] or [He1]). It is a complete seminormed space. In this paper equality of cohomology groups means topological isomorphism of seminormed spaces.

For a Banach space \( E \), we will denote by \( E^* \) the dual space of \( E \). In the case of Banach algebras \( A \), a Banach \( A \)-bimodule \( M = (M^*)^* \), where \( M^* \) is a Banach \( A \)-bimodule, is called dual. Note that \( H^n(A, (M^*)^*) \) can be computed as the \( n \)-dimensional cohomology of the dual complex to \( C^\sim(A, M) \) (see, for example, [Jo] or [He1, II.5.27]). A Banach algebra \( A \) such that \( H^1(A, X^*) = \{0\} \) for every Banach \( A \)-bimodule \( X \) (see [Pat2] and [PaSm]). The weak bidimension of a Banach algebra \( A \) is

\[
db_A = \inf \{ n : H^{n+1}(A, X^*) = 0 \text{ for all } X \in A\text{-mod} \}
\]

(see [Da]). A Banach algebra \( A \) is called simplicially trivial if \( H^n(A, A^*) = \{0\} \) for all \( n \geq 1 \).

3. Excision in continuous cohomology

In this section we prove the following theorem.

**Theorem 3.1.** Let

\[ 0 \to I \xrightarrow{i} A \xrightarrow{j} A/I \to 0 \]

be an extension of Banach algebras and let \( X \) be a Banach \( A \)-bimodule such that \( TX = XT \). Suppose \( I \) has a bounded approximate identity. Then there exists an associated long exact sequence of continuous cohomology groups

\[ 0 \to H^0(A/I, (X/IX)^*) \to \cdots \]

\[ \to H^{n-1}(I, (IX)^*) \to H^n(A/I, (X/IX)^*) \to H^n(A, X^*) \to H^n(I, (IX)^*) \to \cdots. \]

We deduce the theorem from a sequence of lemmas and propositions. To start with we need the following result (cp. [Jo, Proposition 1.8]).

**Lemma 3.2.** Let \( A \) be a Banach algebra, let \( I \) be a left (right) closed ideal of \( A \) and let \( X \) be a left (right) Banach \( A \)-module. Suppose that \( I \) has a left (right) bounded approximate identity \( (e_a)_{a \in A} \). Then the short exact sequence

\[ 0 \to IX \xrightarrow{i} X \xrightarrow{j} X/IX \to 0 \]

\( (0 \to I X \xrightarrow{i} X \xrightarrow{j} X/I X \to 0) \) (3.1)
is weakly admissible.

Proof. Consider the Fréchet filter $F$ on $\Lambda$, with base $\{Q_\lambda : \lambda \in \Lambda\}$, where $Q_\lambda = \{\alpha \in \Lambda : \alpha \geq \lambda\}$. Thus
$$F = \{ E \subset \Lambda : \text{there is a } \lambda \in \Lambda \text{ such that } Q_\lambda \subset E\}.$$ 
Let $U$ be an ultrafilter on $\Lambda$ which refines $F$. One can find information on filters in [Bo].

For $f \in (\text{IX})^*$ we define $g_f \in X^*$ by
$$g_f(x) = \lim_{\alpha \to U} f(i^{-1}(e_{\alpha} \cdot x)) \text{ for all } x \in X.$$ 
It is easy to check that $g_f$ is a bounded linear functional, the operator
$$L : (\text{IX})^* \to X^* : f \mapsto g_f$$
is a bounded linear operator and $i^* \circ L = id_{(\text{IX})^*}$. Thus the dual short sequence of (3.1) is admissible. \hfill $\Box$

**Corollary 3.3.** Let
$$0 \to I \xrightarrow{i} A \xrightarrow{j} A/I \to 0 \quad (3.2)$$
be an extension of Banach algebras. Suppose that $I$ has a left or right bounded approximate identity. Then (3.2) is weakly admissible.

**Remark.** Let $A$ be a Banach algebra and let $X$ be a Banach $A$-bimodule such that $\overline{IX} = X/I$. Suppose that $I$ has a left or right bounded approximate identity. Then, by the Cohen factorization theorem (see, for example, [Pa, 5.2.5]), we have $I = I^2$. Thus it is easy to see that $\overline{IX} = X/I = X IT$.

**Proposition 3.4.** Let $A$ be a Banach algebra and let
$$0 \to X' \xrightarrow{i} X \xrightarrow{j} X'' \to 0$$
be a weakly admissible sequence of left Banach $A$-modules. Then, for any right Banach $A$-module $Y$, there exists in $\text{Ban}$ an exact sequence
$$0 \to \text{Ext}^0_A(X'', Y^*) \to \text{Ext}^0_A(X, Y^*) \to \text{Ext}^1_A(X'', Y^*) \to \cdots$$
$$\cdots \to \text{Ext}^n_A(X'', Y^*) \to \text{Ext}^n_A(X, Y^*) \to \text{Ext}^{n+1}_A(X'', Y^*) \to \cdots$$

Proof. By the assumption, $0 \to (X'')^* \xrightarrow{j^*} X^* \xrightarrow{i^*} (X')^* \to 0$ is admissible. Hence, by [He1, Theorem III.4.4], there exists in $\text{Ban}$ an exact sequence
$$0 \to \text{Ext}^0_{A^{op}}(Y, (X'')^*) \to \text{Ext}^0_{A^{op}}(Y, X^*) \to \text{Ext}^0_{A^{op}}(Y, (X')^*) \to \cdots$$
$$\cdots \to \text{Ext}^n_{A^{op}}(Y, X^*) \to \text{Ext}^n_{A^{op}}(Y, (X')^*) \to \text{Ext}^{n+1}_{A^{op}}(Y, (X'')^*) \to \cdots$$
Now we only have to recall that, for any left Banach $A$-module $Z$ and $n \geq 0$,

$$\text{Ext}_*^n(Z, Y^*) = \text{Ext}_*^{n+1}(Y, Z^*)$$

(see [He1, Prop.III.4.13]).

**Corollary 3.5.** Let $A$ be a Banach algebra, let $I$ be a closed two-sided ideal of $A_+$ and let $X$ be a Banach $A$-bimodule. Suppose that $I$ has a left or right bounded approximate identity $(e_\alpha)_{\alpha \in \Lambda}$. Then the following is a long exact sequence in $\text{Ban}$:

$$0 \rightarrow \text{Ext}_A^0(A/I, X^*) \rightarrow \text{Ext}_A^0(A_+, X^*) \rightarrow \text{Ext}_A^0(I, X^*) \rightarrow \ldots$$

$$\ldots \rightarrow \text{Ext}_A^n(A_+, X^*) \rightarrow \text{Ext}_A^n(I, X^*) \rightarrow \text{Ext}_A^{n+1}(A/I, X^*) \rightarrow \ldots$$

**Proof.** This follows from Corollary 3.3 and Proposition 3.4.

**Lemma 3.6.** Let $A$ be a Banach algebra and let $I$ be a closed two-sided ideal of $A_+$. Suppose that $I$ has a bounded approximate identity $(e_\alpha)_{\alpha \in \Lambda}$. Then the sequence

$$0 \rightarrow I \xrightarrow{\varepsilon} I \hat{\otimes} I \xrightarrow{d_0} I \hat{\otimes} I \rightarrow \ldots \rightarrow I \hat{\otimes}(n+2) \xrightarrow{d_n} I \hat{\otimes}(n+3) \rightarrow \ldots$$

where $\varepsilon(b_0 \otimes b_1) = b_0 b_1$ and

$$d_n(b_0 \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_{n+2}) = \sum_{i=0}^{n+1} (-1)^i (b_0 \otimes b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_{n+2}),$$

is a pseudo-resolution of $I$ in $A$-mod-$A$.

**Proof.** It is easy to check that $d_{n-1} \circ d_n = 0$ for $n \geq 1$ and $\varepsilon \circ d_0 = 0$. Thus (3.3) is a complex. By [He1, Theorem VII.1.5], $I$ is strictly flat as a left and as a right Banach $A$-module. Hence, by [He1, Proposition VII.2.4], for any $n \geq 2$, the Banach $A$-bimodule $I \hat{\otimes} n$ is flat.

Now we have to show that the dual complex

$$0 \rightarrow I^* \xrightarrow{\varepsilon^*} (I \hat{\otimes} I)^* \xrightarrow{d_0^*} (I \hat{\otimes} I)^* \rightarrow \ldots \rightarrow (I \hat{\otimes}(n+2))^* \xrightarrow{d_n^*} (I \hat{\otimes}(n+3))^* \rightarrow \ldots$$

(3.4)

is admissible. We consider the Fréchet filter $F$ on $\Lambda$, with base $\{Q_\lambda : \lambda \in \Lambda\}$, where $Q_\lambda = \{\alpha \in \Lambda : \alpha \geq \lambda\}$. Thus

$$F = \{E \subset \Lambda : \text{there is a } \lambda \in \Lambda \text{ such that } Q_\lambda \subseteq E\}.$$

Let $U$ be an ultrafilter on $\Lambda$ which refines $F$. For $n \geq -1$ and $f \in (I \hat{\otimes}(n+3))^*$, we define $g_f \in (I \hat{\otimes}(n+2))^*$ by

$$g_f(u) = \lim_{\alpha \in U} f(e_\alpha \otimes u) \text{ for all } u \in I \hat{\otimes}(n+2).$$

One can check the following: $g_f$ is a bounded linear functional, the operator

$$s_n : (I \hat{\otimes}(n+3))^* \rightarrow (I \hat{\otimes}(n+2))^* : f \mapsto g_f$$
is a bounded linear operator, $d^n_{n-1} \circ s_{n-1} + s_n \circ d^n_n = id_{(I^O(n+2))}$, for all $n \geq 1$ and $e^* \circ s_{-1} + s_0 \circ d^0_0 = id_{(I^O(1))}$. Thus (3.4) is admissible (see [He1, III.1.9]). Therefore, by definition, (3.3) is a pseudo-resolution of $I$ in $A\text{-mod-}A$. \qed

**Proposition 3.7.** Let $A$ be a Banach algebra, let $I$ be a closed two-sided ideal of $A_+$ and let $X$ be a Banach $A$-bimodule such that $TX = XI$. Suppose that $I$ has a bounded approximate identity $(e_a)_{a \in A}$. Then

$$\mathcal{H}^n(I, (TX)^*) = \text{Ext}^n_A(I, (TX)^*) \quad \text{for all } n \geq 0.$$  

**Proof.** By [He1, VII.1.19], for any Banach $A$-bimodule $Y$, $\text{Ext}^n_A(I, Y^*)$ is the cohomology of the complex $\mathcal{A}h_A(X, Y^*)$ where $0 \leftarrow I \leftarrow X$ is a pseudo-resolution of $I$ in $A\text{-mod-}A$. Thus, by Lemma 3.6, $\text{Ext}^n_A(I, Y^*)$ can be computed by using the pseudo-resolution (3.3). Hence $\text{Ext}^n_A(I, Y^*)$ is the cohomology of the complex

$$0 \rightarrow A h_A(I^{\widehat{\otimes}2}, Y^*) \xrightarrow{h(d_n)} \cdots \rightarrow A h_A(I^{\widehat{\otimes}(n+2)}, Y^*) \xrightarrow{h(d_n)} A h_A(I^{\widehat{\otimes}(n+3)}, Y^*) \rightarrow \ldots.$$  

(3.5)

where $h(d_n)$ is the operator defined by $h(d_n)(\varphi) = \varphi \circ d_n$ for $\varphi \in A h_A(I^{\widehat{\otimes}(n+2)}, Y^*)$.

Now prove that, when $Y = TX$, the last complex is isomorphic to the standard cohomological complex $C^\cdot(I, Y^*)$

$$0 \rightarrow C^0(I, Y^*) \xrightarrow{\delta^0} \cdots \rightarrow C^n(I, Y^*) \xrightarrow{\delta^n} C^{n+1}(I, Y^*) \rightarrow \ldots.$$  

To show this we consider the Fréchet filter $F$ on $\Lambda$ as in Lemma 3.2 and take an ultrafilter $U$ on $\Lambda$ which refines $F$. Further, for $n \geq 0$, one can define the following bounded linear operators:

$$L_n : A h_A(I^{\widehat{\otimes}(n+2)}, (TX)^*) \rightarrow C^n(I, (TX)^*) : \varphi \mapsto f_\varphi$$

by $f_\varphi(b_1, b_2, \ldots, b_n)(y) = \lim_{\alpha \rightarrow U} \varphi(e_\alpha \otimes b_1 \otimes \cdots \otimes b_n) \otimes e_\alpha(y)$ where $b_1, \ldots, b_n \in I$ and $y \in TX$; and

$$G_n : C^n(I, (TX)^*) \rightarrow A h_A(I^{\widehat{\otimes}(n+2)}, (TX)^*) : f \mapsto \varphi_f$$

by $\varphi_f(b_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1})(y) = f(b_1, \ldots, b_n)(b_{n+1} \cdot y \cdot b_0)$ where $b_1 \in I$ and $y \in TX$. It is easy to check that $\varphi_f$ is a morphism of $A$-bimodules. Let $a \in A$; then

$$[a \cdot \varphi_f(b_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1})](y) = \varphi_f(b_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1})(y \cdot a) =$$

$$f(b_1, \ldots, b_n)(b_{n+1} \cdot (y \cdot a) \cdot b_0) = \varphi_f(ab_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1})(y)$$

and

$$[\varphi_f(b_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1}) \cdot a](y) = \varphi_f(b_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1})(a \cdot y) =$$

$$f(b_1, \ldots, b_n)(b_{n+1} \cdot (a \cdot y) \cdot b_0) = \varphi_f(b_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1}a)(y)$$
for all \( a \in A, b_i \in I \) and \( y \in TX \). Now it is routine to check that \( G_n \) is indeed the inverse map of \( L_n \) since \( TX = X/T \) and that
\[
L_n = \{ L_n : A \to (TX)^* \to C^\alpha(I, (TX)^*) \}
\]
is a continuous morphism of the complexes (3.5) and \( C^\alpha(I, (TX)^*) \) in the category of Banach spaces. Thus we have \( H^n(I, (TX)^*) = \text{Ext}_{A^c}^n(I, (TX)^*) \) for all \( n \geq 0 \).

**Lemma 3.8.** Let \( A \) be a Banach algebra, let \( I \) be a closed two-sided ideal of \( A^+ \) and let \( X \) be a Banach \( A \)-bimodule such that \( TX = X/T \). Suppose that \( I \) has a bounded approximate identity. Then
\[
\text{Ext}_{A^c}^n(I, (X/TX)^*) = \{0\} \quad \text{for all } n \geq 0.
\]

**Proof.** As we have shown in Proposition 3.7, for any Banach \( A \)-bimodule \( Y \), \( \text{Ext}_{A^c}^n(I, Y^*) \) is the cohomology of the complex
\[
0 \to A \xrightarrow{h_A(I^\otimes 2, Y^*)} \cdots \xrightarrow{h_A(I^\otimes (n+2), Y^*)} A \xrightarrow{h_A(I^\otimes (n+3), Y^*)} \cdots
\]

Since \( I \) has a bounded approximate identity, we have that \( TX = X/T = X/TX \). For \( Y = X/TX \) we can check that every \( y \in A \) \( h_A(I^\otimes (n+2), Y^*) \), \( n \geq 0 \), has the following property: for \( b_0, \ldots, b_{n+1} \in I \) where \( b_i = b_i^1 b_i^2 \) with \( b_i^1, b_i^2 \in I \) and \( y \in X/TX \),
\[
[\varphi(b_0 \otimes \cdots \otimes b_{n+1})](y) = [\varphi(b_0^1 b_0^2 \otimes \cdots \otimes b_{n+1}^1 b_{n+1}^2)](y) = [b_0^1 \cdot \varphi(b_0^2 \otimes \cdots \otimes b_{n+1}^2) \cdot b_{n+1}^2](y)
\]

Therefore \( h_A(I^\otimes (n+2), (X/TX)^*) = \{0\} \) for all \( n \geq 0 \) and so \( \text{Ext}_{A^c}^n(I, (X/TX)^*) = \{0\} \) for all \( n \geq 0 \).

**Proposition 3.9.** Let \( A \) be a Banach algebra, let \( I \) be a closed two-sided ideal of \( A^+ \) and let \( X \) be a Banach \( A \)-bimodule such that \( TX = X/T \). Suppose that \( A^+/I \) is flat as a left and as a right Banach \( A \)-module (it suffices that \( I \) has a bounded approximate identity). Then
\[
H^n(A/I, (X/TX)^*) = \text{Ext}_{A^c}^n(A^+/I, (X/TX)^*) \quad \text{for all } n \geq 0.
\]

**Proof.** In the case that \( I \) has a bounded approximate identity, by [He1, Theorem VII.1.5], \( A^+/I \) is flat as a left and as a right Banach \( A \)-module. By [He1, Proposition VII.2.4], for any \( n \geq 0 \), \( (A^+/I)^\otimes (n+2) \) is a flat Banach \( A \)-bimodule. Thus one can check that the sequence
\[
0 \to A^+/I \xleftarrow{\varepsilon} A^+/I \otimes A^+/I \xrightarrow{d_0} \cdots \xrightarrow{d_{n+1}} (A^+/I)^\otimes (n+3) \to \cdots, \quad (3.6)
\]
where \( \varepsilon(\bar{a}_0 \otimes \bar{a}_1) = \bar{a}_0 \bar{a}_1 \) and
\[
d_n(\bar{a}_0 \otimes \bar{a}_1 \otimes \bar{a}_2 \otimes \cdots \otimes \bar{a}_{n+2}) = \sum_{i=0}^{n+1} (-1)^i (\bar{a}_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_{i+1} \otimes \cdots \otimes \bar{a}_{n+2}),
\]
is an admissible pseudo-resolution of \( A_+/I \) in \( A\)-mod-\( A \).

By [He1, VII.1.19], for any Banach \( A \)-bimodule \( Y \), \( \text{Ext}^n_{A^\#}(A_+/I, Y^\#) \) is the cohomology of the complex \( A h_A(X, Y^\#) \) where \( 0 \rightarrow A_+/I \rightarrow X \) is a pseudo-resolution of \( A_+/I \) in \( A\)-mod-\( A \). Thus \( \text{Ext}^n_{A^\#}(A_+/I, Y^\#) \) can be computed by using the pseudo-resolution (3.6). Hence \( \text{Ext}^n_{A^\#}(A_+/I, Y^\#) \) is the cohomology of the complex
\[
0 \rightarrow A \rightarrow (A_+/I)^{\otimes 2}, Y^\#) \xrightarrow{h(d_n)} \cdots \rightarrow (A_+/I)^{\otimes (n+2)}, Y^\#) \rightarrow \cdots, \tag{3.7}
\]
where \( h(d_n) \) is the operator defined by \( h(d_n)(\varphi) = \varphi \circ d_n \) for \( \varphi \in A \rightarrow (A_+/I)^{\otimes (n+2)}, Y^\#) \).

Now we prove that, when \( Y = X/\overline{TX} \), the last complex is isomorphic to the standard cohomological complex \( C^\sim(A/I, Y^\#) \),
\[
0 \rightarrow C^0(A_+/I, Y^\#) \xrightarrow{\delta^0} \cdots \rightarrow C^n(A_+/I, Y^\#) \xrightarrow{\delta^n} C^{n+1}(A_+/I, Y^\#) \rightarrow \cdots.
\]
To show this one can define, for \( n \geq 0 \), the bounded linear operators:
\[
L_n : A h_A(A_+/I)^{\otimes (n+2)}, (X/\overline{TX})^\#) \rightarrow C^n(A_+/I, (X/\overline{TX})^\#) : \varphi \mapsto f_\varphi
\]
by \( f_\varphi(\bar{a}_1, \ldots, \bar{a}_n)(y) = \varphi(\bar{e}_+ \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes \bar{e}_+)(y) \) where \( \bar{a}_1, \ldots, \bar{a}_n \in A_+/I \) and \( y \in X/\overline{TX} \), and
\[
G_n : C^n(A_+/I, (X/\overline{TX})^\#) \rightarrow A h_A(A_+/I)^{\otimes (n+2)}, (X/\overline{TX})^\#) : f \mapsto \varphi_f
\]
by \( \varphi_f(\bar{a}_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes \bar{a}_{n+1})(y) = f(\bar{a}_1, \ldots, \bar{a}_n)(\bar{a}_{n+1} \otimes \bar{a}_0) \) where \( \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{n+1} \in A_+/I \) and \( y \in Y \). One can check that \( \varphi_f \) is a morphism of \( A \)-bimodules.

It is routine to show that \( G_n \) is indeed the inverse map of \( L_n \) and that \( L_n = \{ L_n : A h_A(A_+/I)^{\otimes (n+2)}, (X/\overline{TX})^\#) \rightarrow C^n(A_+/I, (X/\overline{TX})^\#) \} \) is a continuous morphism of the complexes (3.7) and \( C^\sim(A_+/I, (X/\overline{TX})^\#) \) in the category of Banach spaces. Thus in view of [He1, Theorem III.4.9] we have \( H^n(A/I, (X/\overline{TX})^\#) = \text{Ext}^n_{A^\#}(A_+/I, (X/\overline{TX})^\#) \) for all \( n \geq 0 \).

**Lemma 3.10.** Let \( A \) be a Banach algebra, let \( I \) be a closed two-sided ideal of \( A_+ \) and let \( X \) be a Banach \( A \)-bimodule such that \( TX = XT = TXT \). Suppose that \( A_+/I \) is flat as a left and as a right Banach \( A \)-module. Then
\[
\text{Ext}^n_{A^\#}(A_+/I, (TX)^\#) = \{ 0 \} \text{ for all } n \geq 0.
\]

**Proof.** By [He1, Proposition III.4.13], for any Banach \( A \)-bimodule \( Y \),
\[
\text{Ext}^n_{A^\#}(A_+/I, Y^\#) = \text{Ext}^n_{A^\#}(Y, (A_+/I)^*)
\]
for all \( n \geq 0 \). The groups \( \text{Ext}^n_{A^\#}(Y, (A_+/I)^*) \) can be computed by using the following injective resolution of \( (A_+/I)^* \) in \( A\)-mod-\( A \),
that is, the dual of the admissible pseudo-resolution (3.6) of $A_+/I$ in $A\text{-mod} A$: 

$$0 \rightarrow (A_+/I)^* \xrightarrow{\varepsilon} (A_+/I \otimes A_+/I)^* \xrightarrow{d_2^A} \ldots$$

$$\ldots \rightarrow ((A_+/I)^{\hat{\otimes}(n+2)})^* \xrightarrow{d_2^A} ((A_+/I)^{\hat{\otimes}(n+3)})^* \rightarrow \ldots.$$ 

Hence $\text{Ext}_{A^*\text{-mod}}^{n}(Y, (A_+/I)^*)$ is the cohomology of the following complex $0 \rightarrow A h_A(Y, ((A_+/I)^{\hat{\otimes}(n+2)})^*) \xrightarrow{\text{h}(d_2^A)} \ldots$

$$\ldots \rightarrow A h_A(Y, ((A_+/I)^{\hat{\otimes}(n+2)})^*) \xrightarrow{\text{h}(d_2^A)} A h_A(Y, ((A_+/I)^{\hat{\otimes}(n+3)})^*) \rightarrow \ldots,$$

where $\text{h}(d_2^A)$ is the operator defined by $\text{h}(d_2^A)(\varphi) = d_2^A \circ \varphi$ for $\varphi \in A h_A(Y, ((A_+/I)^{\hat{\otimes}(n+2)})^*)$.

For $Y = TXI$, we can see that every $\varphi \in A h_A(Y, ((A_+/I)^{\hat{\otimes}(n+2)})^*)$, $n \geq 0$, has the following property:

$$[\varphi(b_1 \cdot x \cdot b_2)][\tilde{a}_0 \otimes \cdots \otimes \tilde{a}_{n+1}] = [b_1 \cdot \varphi(x) \cdot b_2][\tilde{a}_0 \otimes \cdots \otimes \tilde{a}_{n+1}] =$$

$$\varphi(x)(b_2 \cdot \tilde{a}_0 \otimes \cdots \otimes \tilde{a}_{n+1} \cdot b_1) = 0$$

where $b_1, b_2 \in I$, $x \in X$ and $\tilde{a}_0, \ldots, \tilde{a}_{n+1} \in A_+/I$. Therefore $A h_A(TXI, ((A_+/I)^{\hat{\otimes}(n+2)})^*) = \{0\}$ for all $n \geq 0$ and so $\text{Ext}_{A^*\text{-mod}}^{n}(A_+/I, (TXI)^*) = \text{Ext}_{(A^*\text{-mod})^{n}}^{n}(TXI, (A_+/I)^*) = \{0\}$ for all $n \geq 0$. 

Corollary 3.11. Let $A$ be a Banach algebra, let $I$ be a closed two-sided ideal of $A_+$ and let $X$ be a Banach $A$-bimodule such that $TX = XI$. Suppose that $I$ has a bounded approximate identity. Then $\text{Ext}_{A^*\text{-mod}}^{n}(A_+/I, (TX)^*) = \{0\}$ for all $n \geq 0$.

Proof. Since $I$ has a bounded approximate identity, we have that $TX = XI = TXI$. By [He1, Theorem VII.1.5] $A_+/I$ is flat as a left and as a right Banach $A$-module. Therefore the result follows from Lemma 3.10. 

Proposition 3.12. Let $A$ be a Banach algebra, let $I$ be a closed two-sided ideal of $A_+$ and let $X$ be a Banach $A$-bimodule such that $TX = XI$. Suppose that $I$ has a bounded approximate identity. Then, for all $n \geq 0$,

$$\text{Ext}_{A^*\text{-mod}}^{n}(I, X^*) = \text{Ext}_{A^*\text{-mod}}^{n}(I, (TX)^*) = \mathcal{H}^n(I, (TX)^*)$$

and

$$\text{Ext}_{A^*\text{-mod}}^{n}(A_+/I, X^*) = \text{Ext}_{A^*\text{-mod}}^{n}(A_+/I, (X/IX)^*) = \mathcal{H}^n(A/I, (X/IX)^*).$$

Proof. By Lemma 3.2, $0 \rightarrow TX \xrightarrow{j} X \xrightarrow{\delta} X/IX \rightarrow 0$ is weakly admissible. Therefore, by [He1, Theorem III.4.4], the following long exact sequences exist in $\text{Ban}$:

$$0 \rightarrow \text{Ext}_{A^*\text{-mod}}^{0}(I, (X/IX)^*) \rightarrow \text{Ext}_{A^*\text{-mod}}^{0}(I, X^*) \rightarrow \ldots$$
\[ \cdots \rightarrow \text{Ext}^n_{A^e}(I, X^*) \rightarrow \text{Ext}^n_{A^e}(I, (IX)^*) \rightarrow \text{Ext}^{n+1}_{A^e}(I, (X/IX)^*) \rightarrow \cdots \]

and

\[ 0 \rightarrow \text{Ext}^0_{A^e}(A_+/I, (X/IX)^*) \rightarrow \text{Ext}^0_{A^e}(A_+/I, X^*) \rightarrow \cdots \]

Thus, in view of Lemma 3.8, we have \( \text{Ext}^n_{A^e}(I, X^*) = \text{Ext}^n_{A^e}(I, (IX)^*) \) for all \( n \geq 0 \).

In view of Corollary 3.11, we deduce that \( \text{Ext}^n_{A^e}(A_+/I, X^*) = \text{Ext}^n_{A^e}(A_+/I, (X/IX)^*) \) for all \( n \geq 0 \). The result follows from Propositions 3.7 and 3.9.

**End of the proof of Theorem 3.1.** By Corollary 3.5, there is in \( \text{Ban} \) a long exact sequence

\[ 0 \rightarrow \text{Ext}^0_{A^e}(A_+/I, X^*) \rightarrow \text{Ext}^0_{A^e}(A_+/I, X^*) \rightarrow \text{Ext}^0_{A^e}(A_+/I, X^*) \rightarrow \cdots \]

\[ \cdots \rightarrow \text{Ext}^n_{A^e}(A_+/I, X^*) \rightarrow \text{Ext}^n_{A^e}(A_+/I, X^*) \rightarrow \text{Ext}^{n+1}_{A^e}(A_+/I, X^*) \rightarrow \cdots \]

By [He1, Theorem III.4.9], \( \text{Ext}^n_{A^e}(A_+/I, X^*) = \mathcal{H}^n(A, X^*) \) for all \( n \geq 0 \). By Proposition 3.12, \( \text{Ext}^n_{A^e}(A_+/I, X^*) = \mathcal{H}^n(I, (IX)^*) \) and \( \text{Ext}^n_{A^e}(A_+/I, X^*) = \mathcal{H}^n(A/I, (X/IX)^*) \) for all \( n \geq 0 \). Thus the theorem is proved.

**4. Connections between excision in homology and cohomology of Banach algebras**

In this section we prove (Theorem 4.4) that a weakly admissible extension of Banach algebras has the excision property in continuous cohomology groups \( \mathcal{H}^n(A, X^*) \) if and only if it has the same property in continuous homology groups \( \mathcal{H}_n(A, X) \). For this purpose we need to adapt some algebraic arguments to the continuous case and check certain properties of the tensor product of operators. To start with we need the following result.

**Proposition 4.1.** [Ly2, Proposition 3.2] Let \( K_\sim \) and \( P_\sim \) be chain complexes of Banach spaces and continuous linear operators. Suppose there is a continuous morphism of chain complexes \( \psi_\sim : K_\sim \rightarrow P_\sim \) such that, for each \( n \), \( \psi_n \) is injective and \( \text{Im} \psi_n \) is closed. Then the following conditions are equivalent:

(i) \( \psi_\sim \) is a topological quasi-isomorphism of the chain complexes \( K_\sim \) and \( P_\sim \);

(ii) \( \psi_n^\sim \) is a topological quasi-isomorphism of the cochain, dual complexes \( K^\sim_n \) and \( P^\sim_n \).

The following lemma is widely known (see, for example, [Ly2, Lemma 3.3]).

**Lemma 4.2.** Let

\[ 0 \rightarrow Y \xrightarrow{i} Z \xrightarrow{j} W \rightarrow 0 \]
be a weakly admissible sequence of Banach spaces and continuous operators. Then, for every Banach space $X$, the sequence

$$0 \to X \hat{\otimes} Y \xrightarrow{id \otimes i} X \hat{\otimes} Z \xrightarrow{id \otimes j} X \hat{\otimes} W \to 0$$

is weakly admissible.

As to the projective tensor product of Banach spaces, the difficulty lies in the well known fact that the tensor product of two injective operators need not be injective. To circumvent this difficulty we prove the following lemma.

**Lemma 4.3.** Let

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0 \quad \text{and} \quad 0 \to X' \xrightarrow{s} X \xrightarrow{t} X'' \to 0$$

be weakly admissible sequences of Banach spaces and continuous operators. Then, for every $n \geq 0$, the following conditions are satisfied

- (i) the operator

  $$s \hat{\otimes} \cdots \hat{\otimes} i : X' \hat{\otimes} Y \hat{\otimes} \cdots \hat{\otimes} Y \to X \hat{\otimes} Z \hat{\otimes} \cdots \hat{\otimes} Z$$

  is injective and $\text{Im} \ (s \hat{\otimes} \cdots \hat{\otimes} i)$ is closed;

- (ii) the complex

  $$0 \to X' \hat{\otimes} Y \hat{\otimes} \cdots \hat{\otimes} i \hat{\otimes} X' \hat{\otimes} \cdots \hat{\otimes} X' \hat{\otimes} \cdots \hat{\otimes} X' \hat{\otimes} W \hat{\otimes} \cdots \hat{\otimes} W \xrightarrow{s \hat{\otimes} \cdots \hat{\otimes} j} X'' \hat{\otimes} W \hat{\otimes} \cdots \hat{\otimes} W \to 0$$

  is exact at $X' \hat{\otimes} Y \hat{\otimes} \cdots \hat{\otimes} X' \hat{\otimes} W \hat{\otimes} \cdots \hat{\otimes} W$.

**Proof.** (i) We can see that

$$s \hat{\otimes} \cdots \hat{\otimes} i = \kappa_{n+1} \circ (id_{Z \hat{\otimes} n} \otimes s) \circ \kappa_n \circ (id_{Z \hat{\otimes} (n-1)} \otimes i) \circ \kappa_{n-1} \circ \cdots \circ (id_{Z \hat{\otimes} (n-2)} \otimes i) \circ \kappa_1 \circ (id_{X' \hat{\otimes} Y \hat{\otimes} n} \otimes i),$$

where $\kappa_i : Z^{\otimes (i-1)} \otimes X' \hat{\otimes} Y \hat{\otimes} (n-i) \otimes Z \equiv Z^{\otimes i} \otimes X' \hat{\otimes} Y \hat{\otimes} (n-i), \ i = 1, \ldots, n$, and $\kappa_{n+1} : Z^{\otimes n} \otimes X \equiv X \hat{\otimes} Z^{\otimes n}$ are given by

$$\kappa_i(x_0 \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_0 \otimes x_0 \otimes \cdots \otimes x_{n-1}, \ i = 1, \ldots, n + 1.$$

Thus, by Lemma 4.2, $(s \hat{\otimes} \cdots \hat{\otimes} i)^* $ is surjective as the composition of surjective operators. Therefore, by [Ed; 8.6.15], $s \hat{\otimes} \cdots \hat{\otimes} i$ is injective and $\text{Im} \ (s \hat{\otimes} \cdots \hat{\otimes} i)$ is closed in $X \hat{\otimes} Z^{\otimes n}$.

(ii) It follows from (i) and the fact that the projective tensor product of surjective operators is also surjective. \(\square\)

**Theorem 4.4.** Let

$$0 \to I \xrightarrow{i} A \xrightarrow{j} A/I \to 0 \quad (4.1)$$
be a weakly admissible extension of Banach algebras and let $X$ be a Banach $A$-bimodule such that $\overline{TX} = X/TX$. Suppose that the short exact sequence

$$0 \to \overline{TX} \xrightarrow{i} X \xrightarrow{\pi} X/\overline{TX} \to 0 \quad (4.2)$$

is weakly admissible. Then there exists an associated long exact sequence of continuous homology groups

$$\cdots \to \mathcal{H}_n(I, \overline{TX}) \to \mathcal{H}_n(A, X) \to \mathcal{H}_n(A/I, X/\overline{TX}) \to \mathcal{H}_{n-1}(I, \overline{TX}) \to \cdots \quad (4.3)$$

if and only if there exists an associated long exact sequence of continuous cohomology groups

$$0 \to \mathcal{H}^n(A/I, X/\overline{TX})^* \to \cdots \to \mathcal{H}^n(A, X)^* \to \mathcal{H}^n(A/I, X/\overline{TX})^* \to \cdots \quad (4.4)$$

Proof. By Lemma 4.3 (ii), for $n = 0, 1, \ldots$, the extension (4.1) induces a complex of Banach spaces and continuous operators

$$0 \to C_n(I, \overline{TX}) \xrightarrow{s^*} C_n(A, X) \xrightarrow{t^*} C_n(A/I, X/\overline{TX}) \to 0$$

which is exact at $C_n(A/I, X/\overline{TX})$ and $C_n(I, \overline{TX})$. It is routine to check that the diagram

$$\begin{array}{ccc}
0 & \to & 0 \\
\uparrow & & \uparrow \\
0 & \to & C_0(I, \overline{TX}) \xrightarrow{s^*} C_0(A, X) \xrightarrow{t^*} C_0(A/I, X/\overline{TX}) \to 0 \\
\uparrow d_0 & & \uparrow d_0 \\
0 & \to & C_1(I, \overline{TX}) \xrightarrow{s^*} C_1(A, X) \xrightarrow{t^*} C_1(A/I, X/\overline{TX}) \to 0 \\
\uparrow d_1 & & \uparrow d_1 \\
\cdots & \cdots & \cdots \\
\uparrow & \uparrow & \uparrow \\
0 & \to & C_n(I, \overline{TX}) \xrightarrow{s^*} C_n(A, X) \xrightarrow{t^*} C_n(A/I, X/\overline{TX}) \to 0 \\
\uparrow d_n & & \uparrow d_n \\
0 & \to & C_{n+1}(I, \overline{TX}) \xrightarrow{s^*} C_{n+1}(A, X) \xrightarrow{t^*} C_{n+1}(A/I, X/\overline{TX}) \to 0 \\
\uparrow & \uparrow & \uparrow \\
\cdots & \cdots & \cdots \\
\end{array}$$

is commutative. The same is true for cochains. Thus the extension (4.1) induces a short exact sequence of chain complexes

$$0 \to C_-(A; I; X) \to C_-(A, X) \to C_-(A/I, X/\overline{TX}) \to 0$$
and a short exact sequence of cochain complexes

$$0 \to C^\ast(A/I, (X/TX)*) \to C^\ast(A, X^*) \to C^\ast(A; I; X) \to 0$$

in the category of Banach spaces and continuous operators, where $C_\ast(A; I; X)$ is the subcomplex $\text{Ker}(t \otimes j \otimes \ldots \otimes j)$ of $C_\ast(A, X)$; $C^\ast(A; I; X)$ is the complex $[\text{Ker}(t \otimes j \otimes \ldots \otimes j)]^\ast$. Hence, by [He1, Section 0.5.4], there exist a long exact sequence of homology groups

$$\cdots \to \mathcal{H}_{n+1}(A/I, X/TX) \to \mathcal{H}_n(C_\ast(A; I; X)) \to \mathcal{H}_n(A, X) \to \mathcal{H}_n(A/I, X/TX) \to \cdots$$

and a long exact sequence of cohomology groups

$$\cdots \to H^{n-1}(C^\ast(A; I; X)) \to H^n(A, X^*) \to H^n(C^\ast(A; I; X)) \to \cdots.$$ 

By Lemma 4.3 (i),

$$s \otimes t \otimes \ldots \otimes n : TX \otimes I \otimes n \to \text{Ker}(t \otimes j \otimes \ldots \otimes j)$$

is injective and $\text{Im} (s \otimes t \otimes \ldots \otimes n)$ is closed. Therefore, by Proposition 4.1, for the two complexes $C_\ast(I, TX)$ and $C_\ast(A; I; X)$ and the continuous morphism $\{s \otimes t \otimes \ldots \otimes n\}$ of complexes, we have $H_n(C_\ast(A; I; X)) \cong H_n(I, TX)$ for every $n = 0, 1, \ldots$ if and only if $H^n(C^\ast(A; I; X)) \cong H^n(I, (TX)^*)$ for every $n = 0, 1, \ldots$. The result follows from the five lemma [Ma, Lemma 1.3.3].

**Theorem 4.5.** Let $0 \to I \to A \to A/I \to 0$ be an extension of Banach algebras such that $I$ has a bounded approximate identity and let $X$ be a Banach $A$-bimodule $X$ such that $TX = TX$. Then associated long exact sequences of continuous homology groups (4.3) and continuous cohomology groups (4.4) exist in $\text{Ban}$. 

**Proof.** By Lemma 3.2 and Corollary 3.3, the extension of Banach algebras (4.1) and the extension of Banach $A$-bimodules (4.2) are weakly admissible. Thus the result follows from Theorems 3.1 and 4.4. 

**Proposition 4.6.** Let $A$ be a Banach algebra and let

$$0 \to X' \xrightarrow{\delta} X \xrightarrow{\ell} X'' \to 0$$

be a short weakly admissible sequence of Banach $A$-bimodules. Then there exists in $\text{Ban}$ an associated long exact sequence

$$\cdots \to \mathcal{H}_n(A, X') \to \mathcal{H}_n(A, X) \to \mathcal{H}_n(A, X'') \to \mathcal{H}_{n-1}(A, X') \to \cdots$$

(4.6)

$$\cdots \to \mathcal{H}_0(A, X'') \to 0.$$

**Proof.** By Lemma 4.2, for any $n \geq 0$, the sequence

$$0 \to X' \otimes A \otimes n \xrightarrow{s \otimes \text{id}} X \otimes A \otimes n \xrightarrow{t \otimes \text{id}} X'' \otimes A \otimes n \xrightarrow{t \otimes \text{id}} X' \otimes A \otimes n \to 0$$

is exact. Hence, the sequence

$$0 \to \mathcal{H}_0(A, X'') \to \cdots \to \mathcal{H}_n(A, X) \to \mathcal{H}_n(A, X'') \to \mathcal{H}_{n-1}(A, X) \to \cdots$$

is also exact. Therefore, the result follows from Theorem 4.5.
is weakly admissible. One can check that the extension of Banach $A$-bimodules (4.5) induces a short exact sequence of chain complexes

$$0 \to C_\sim(A, X') \to C_\sim(A, X) \to C_\sim(A, X'') \to 0$$

in the category of Banach spaces and continuous operators. Hence, by [He1, Theorem 0.5.7], there exist a long exact sequence of homology groups (4.6).

5. Applications to $n$-amenability of Banach algebras

Recall Proposition 5.1 of [Jo], which asserts that a quotient algebra of an amenable algebra is amenable and that an extension of an amenable algebra by an amenable bi-ideal is an amenable algebra (this can also be found in Corollary 35 and Proposition 39 of [He3]). We prove (Corollary 5.3) that the $n$-amenability of a Banach algebra $A$ implies the $n$-amenability of the two Banach algebras $A/I$ and $I$ when $I$ is a closed two-sided ideal with a bounded approximate identity. We also give some additional information about connections between the (co)homology of a Banach algebra $A$ and the (co)homology of Banach algebras $A/I$ and $I$. The following result on cohomology groups is essentially given in [Jo, Proposition 1.8].

**Lemma 5.1.** Let $A$ be a Banach algebra and let $X$ be a Banach $A$-bimodule. Suppose that $A$ has a bounded approximate identity. Then

$$H_n(A, X) = H_n(A, AX) = H_n(A, AXA)$$

for all $n \geq 1$ and

$$H^n(A, X^*) = H^n(A, (AX)^*) = H^n(A, (AXA)^*)$$

for all $n \geq 1$.

**Proof.** By Lemma 3.2, the short exact sequence of Banach $A$-bimodules

$$0 \to AX \xrightarrow{\phi} X \xrightarrow{\pi} X/AX \to 0$$

is weakly admissible. Thus the sequence

$$0 \to (X/AX)^* \xrightarrow{\pi^*} X^* \xrightarrow{\phi^*} (AX)^* \to 0$$

is admissible. Therefore, by [Jo, Proposition 1.7], [He1, III.4.11] and Proposition 4.6, there exist in $\text{Ban}$ two long exact sequences

$$\cdots \to H_{n+1}(A, X/AX) \to H_n(A, AX) \to H_n(A, X) \to H_n(A, X/AX) \to \cdots$$

(5.1)

and

$$\cdots \to H^n(A, (X/AX)^*) \to H^n(A, AX)^* \to H^n(A, X)^* \to H^{n+1}(A, (X/AX)^*) \to \cdots$$

(5.2)
Note that \( X/\overline{AX} \) is a left annihilator Banach \( A \)-bimodule, and by assumption \( A \) has a bounded approximate identity. Thus by [Jo, Proposition 1.5], \( \mathcal{H}^n(A, (X/\overline{AX})^*) = \{0\} \) for all \( n \geq 1 \), and by [Jo, Corollary 1.3], the homology groups \( \mathcal{H}_n(A, X/\overline{AX}) = \{0\} \) for all \( n \geq 1 \). Hence it follows from the exactness of (5.1) and (5.2) that \( \mathcal{H}_n(A, X) = \mathcal{H}_n(A, AX) \) and \( \mathcal{H}^n(A, X^*) = \mathcal{H}^n(A, (AX)^*) \) for all \( n \geq 1 \).

Similar one can deduce that \( \mathcal{H}_n(A, \overline{AX}) = \mathcal{H}_n(A, AXA) \) and \( \mathcal{H}^n(A, (AXA)^*) = \mathcal{H}^n(A, (AXA)^*) \) for all \( n \geq 1 \).

By the Cohen factorization theorem, \( I \rightarrow H \), associated long exact sequence of continuous cohomology groups \( \cdots \rightarrow \mathcal{H}^{n-1}(I, (\overline{TX})^*) \rightarrow \mathcal{H}^n(I, (\overline{TX})^*) \rightarrow \mathcal{H}^n(I, X^*) \rightarrow \mathcal{H}^n(I, (\overline{TX})^*) \rightarrow \cdots \).

For the homology groups, the result follows from Theorem 4.5 and Lemma 5.1 by the same arguments.

**Proposition 5.2.** Let \( A \) be a Banach algebra and let \( I \) be a closed two-sided ideal of \( A \) and let \( Y \) be a Banach \( A/I \)-bimodule. It is natural to consider \( Y^* \) as a Banach \( A \)-bimodule with multiplications: \( a \cdot y = \bar{a} \cdot y \) and \( y \cdot a = y \cdot \bar{a} \) where \( \bar{a} = a + I \) for all \( a \in A \), \( y \in Y \).

(i) By Theorem 3.1, for the Banach \( A \)-bimodule \( X = ITI \), there exists an associated long exact sequence of continuous cohomology groups

\[ \cdots \rightarrow \mathcal{H}^{n-1}(I, (\overline{TX})^*) \rightarrow \mathcal{H}^n(I, (\overline{TX})^*) \rightarrow \mathcal{H}^n(I, X^*) \rightarrow \mathcal{H}^n(I, (\overline{TX})^*) \rightarrow \cdots \]

By the Cohen factorization theorem, \( I = I^2 \). Hence one can see that \( X = \overline{TX} = TTI \), and so \( X/\overline{TX} = \{0\} \). Thus \( \mathcal{H}^n(A/I, (X/\overline{TX})^*) = \{0\} \) for all \( n \geq 0 \). Therefore, \( \mathcal{H}^n(A, X^*) = \mathcal{H}^n(I, (\overline{TX})^*) \) for all \( n \geq 0 \). Hence by Lemma 5.1,

\[ \mathcal{H}^n(I, Z^*) = \mathcal{H}^n(I, (\overline{TX})^*) = \mathcal{H}^n(I, (\overline{TY})^*) = \mathcal{H}^n(A, X^*) \]

for all \( n \geq 1 \).

(ii) By Theorem 3.1, for the Banach \( A \)-bimodule \( Y \), there exists an associated long exact sequence of continuous cohomology groups

\[ \cdots \rightarrow \mathcal{H}^{n-1}(I, (\overline{TY})^*) \rightarrow \mathcal{H}^n(I, (\overline{TY})^*) \rightarrow \mathcal{H}^n(I, Y^*) \rightarrow \mathcal{H}^n(I, (\overline{TY})^*) \rightarrow \cdots \]

Note that \( \overline{TY} = \{0\} \), and so \( Y/\overline{TY} = Y \). Thus \( \mathcal{H}^n(I, (\overline{TY})^*) = \{0\} \) for all \( n \geq 0 \). Therefore, \( \mathcal{H}^n(A/I, Y^*) = \mathcal{H}^n(A, Y^*) \) for all \( n \geq 0 \).

For the homology groups, the result follows from Theorem 4.5 and Lemma 5.1 by the same arguments.

**Corollary 5.3.** Let \( A \) be a Banach algebra and let \( I \) be a closed two-sided ideal of \( A \). Suppose that \( I \) has a bounded approximate identity. Then

(i) the \( n \)-amenability of \( A \) implies the \( n \)-amenability of the two Banach algebras \( A/I \) and \( I \).
(ii) $db_w A \geq db_w I$;
(iii) $db_w A \geq db_w A/I$.

**Corollary 5.4.** Let $A$ be a biprojective Banach algebra and let $I$ be a closed two-sided ideal of $A$. Suppose that $I$ has a bounded approximate identity. Then $db_w A/I \leq 2$ and $db_w I \leq 2$.

**Proof.** By [He1, Theorem V.2.28], $db_w A \leq 2$. The result follows from Corollary 5.3. □

We also obtain from Theorem 4.5 another way to prove the excision property in continuous simplicial cohomology $H^n(A, A^*)$ and homology $H_n(A, A)$ groups under an additional condition on an ideal (cp. [Ly2, Theorem 4.2]).

**Proposition 5.5.** Let $0 \to I \to A \to A/I \to 0$ be an extension of Banach algebras. Suppose $I$ has a bounded approximate identity. Then there exist associated long exact sequences of continuous simplicial homology groups

$$
\cdots \to \mathcal{H}_n(I, I) \to \mathcal{H}_n(A, A) \to \mathcal{H}_n(A/I, A/I) \to \mathcal{H}_{n-1}(I, I) \to \cdots
$$

and continuous simplicial cohomology groups

$$
\cdots \to \mathcal{H}^{n-1}(I, I^*) \to \mathcal{H}^n(A/I, (A/I)^*) \to \mathcal{H}^n(A, A^*) \to \mathcal{H}^n(I, I^*) \to \cdots
$$

**Proof.** By the Cohen factorization theorem, $I = I^2$. Thus it is easy to see that $I^2 \subseteq IA \subseteq I = I^2$ and $I^2 \subseteq AI \subseteq I = I^2$. The result follows from Theorem 4.5 for the Banach $A$-bimodule $X = A$. □

There is thus a close connection between the simplicial triviality of the Banach algebras $A$, $A/I$ and $I$.

**Corollary 5.6.** Let $A$ be a Banach algebra and let $I$ be a closed two-sided ideal of $A$. Suppose that $I$ has a bounded approximate identity. Then

(i) suppose that $A$ is simplicially trivial; then

$H_{n+1}(A/I, A/I) = H_n(I, I)$ and $H^{n+1}(A/I, (A/I)^*) = H^n(I, I^*)$ for all $n \geq 1$;

(ii) the simplicial triviality of $A/I$ and $I$ implies the simplicial triviality of $A$.

**References**


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