SIMPLICIAL COHOMOLOGY OF SOME SEMIGROUP ALGEBRAS

F. GOURDEAU, A. POURABBAS, AND M. C. WHITE

Abstract. In this paper first we show that $H^3(\ell_1(S), \ell_\infty(S)) = 0$, where $S$ is a semilattice. Next we show that the second order simplicial cohomology of various classes of semigroup algebras is a Banach space.

1. Introduction

In this paper, we investigate the higher simplicial cohomology groups of the convolution algebra $\ell^1(S)$ for various semigroups $S$. Our results are of two types: in some cases, we show that some cohomology groups are 0, while in some other cases, we show that some cohomology groups are Banach spaces. In [3] second order cohomology groups of some semigroup Banach algebras were determined.

For the semilattice $S$ (that is, a commutative semigroup $S$ in which $e^2 = e$ for each $e \in S$) Dales and Duncan [3] showed that $H^2(\ell^1(S), X) = 0$ for any Banach $\ell^1(S)$-module $X$. In this paper we show that $H^3(\ell^1(S), X)$ for any Banach $\ell^1(S)$-module $X$ is a Banach space. In particular we show that the third simplicial cohomology group of $\ell^1(S)$ vanishes.

For the semigroup $\mathbb{Z}_+$, we know [4] that all simplicial cohomology groups of $\ell^1(\mathbb{Z}_+)$ vanish for $n \geq 2$. If we consider the semigroup of integers $\mathbb{N}_a = \{n \in \mathbb{Z}_+ : n \geq a\}$ where $a > 0$, the situation becomes more complicated. In fact, all we are able to show in the general case is that $H^2(\ell^1(\mathbb{N}_a), \ell^\infty(\mathbb{N}_a))$ is a Banach space. This is shown by considering approximately additive functions in Section 3.

Finally in Section 4 and Section 5 we will show that the second simplicial cohomology group of $\ell^1(S)$ is a Banach space, where $S$ is a Clifford semigroup or is a Rees semigroup.

The general idea for showing that a cohomology group is a Banach space is as follow. Let $\delta : C^n(A, X) \rightarrow C^{n+1}(A, X)$ be the boundary map. Then $H^n(A, X)$ is a Banach space if and only if the range of $\delta$ is closed, which is the case if and only if $\delta$ is open onto its range, that is there exists a constant $K$ such that if $\psi = \delta(\phi)$ is such that $\|\psi\| < 1$ then there exists $\phi_1 \in C^n(A, X)$ such that $\|\phi_1\| < K$ and $\psi = \delta(\phi_1)$. This is in turn equivalent to the existence of $\phi_0 \in \ker \delta$ such that $\|\delta(\phi_0)\| < K$ (where $\phi_0 = \phi - \phi_1$).

We now recall some basic results and introduce our notation. Let $A$ be a Banach algebra and let $A'$ be a Banach $A$-bimodule in the usual way. An $n$-cochain is a bounded $n$-linear map $T$ from $A$ to $A'$, which we denote by $T \in C^n(A, A')$. The
map $\delta^n : C^n(A, A') \to C^{n+1}(A, A')$ is defined by

$$ (\delta^n T)(a_1, \ldots, a_{n+1})(a_0) = T(a_2, a_3, \ldots, a_{n+1})(a_0 a_1) - T(a_1 a_2, a_3, \ldots, a_{n+1})(a_0) + T(a_1, a_2 a_3, a_4, \ldots, a_{n+1})(a_0) + \ldots + (-1)^n T(a_1, \ldots, a_{n-1}, a_n a_{n+1})(a_0) + (-1)^{n+1} T(a_1, \ldots, a_n)(a_{n+1} a_0). $$

The $n$-cochain $T$ is an $n$-cocycle if $\delta^n T = 0$ and it is an $n$-coboundary if $T = \delta^{n-1} S$ for some $S \in C^{n-1}(A, A')$. The linear space of all $n$-cocycles is denoted by $Z^n(A, A')$, and the linear space of all $n$-coboundaries is denoted by $B^n(A, A')$. We also recall that $B^n(A, A')$ is included in $Z^n(A, A')$ and that the $n$th simplicial cohomology group $\mathcal{H}^n(A, A')$ is defined by the quotient

$$ \mathcal{H}^n(A, A') = Z^n(A, A') / B^n(A, A'). $$

The $n$-cochain $T$ is called cyclic if $T(a_1, a_2, \ldots, a_n)(a_0) = (-1)^n T(a_0, a_1, \ldots, a_n)(a_0)$, and we denote the linear space of all cyclic $n$-cochains by $C^n(A, A')$. It is well known (see [6]) that the cyclic cochains $C^n(A, A')$ form a subcomplex of $C^n(A, A')$, that is $\delta^n : C^n(A, A') \to C^{n+1}(A, A')$, and so we have cyclic versions of the spaces defined above, which we denote by $\mathcal{B}C^n(A, A')$, $\mathcal{S}C^n(A, A')$ and $\mathcal{H}C^n(A, A')$. Note that it is usual to denote the cyclic cohomology group by $\mathcal{H}C^n(A, A')$, as there is only one bimodule used, namely $A'$. For the same reason, we will often denote $C^n(A, A')$ by $\mathcal{C}C^n(A, A')$, $\mathcal{B}C^n(A, A')$ by $\mathcal{B}C^n(A)$ and $\mathcal{S}C^n(A, A')$ by $\mathcal{S}C^n(A)$.

Let $X$ be a Banach $\ell^1(S)$-module. As usual, we identify the element of $S$ with point masses in $\ell^1(S)$. There is an obvious one-to-one correspondence between bounded $n$-cochain $\phi \in \mathcal{C}^n(\ell^1(S), X)$ and the bounded function from $S \times \ldots \times S$ into $X$. Thus we use the same notation for $\phi \in \mathcal{C}^n(\ell^1(S), X)$ and $\phi$ as a function on $S \times \ldots \times S$.

We shall use a much simplified version of the main results of [9] and present it in the following theorem:

**Theorem 1.1.** [9] Let $G$ be a discrete group, and let $S$ be a $G$-set. Then for every $\psi \in \mathcal{C}^1(\ell^1(G), \ell^\infty(S))$ there exist a $\tilde{\psi} \in \mathcal{C}^1(\ell^1(G), \ell^\infty(S))$ such that $\delta \tilde{\psi} = \delta \psi$ and $\|\tilde{\psi}\| \leq 2 \|\delta \psi\|$, that means $\mathcal{H}^1(\ell^1(G), \ell^\infty(S))$ is a Banach space.

### 2. Semilattice Algebra

The semigroup $S$ is called a semilattice if $S$ is a commutative semigroup such that $c^2 = c$ for every $c \in S$. In this section, we show that for such semigroup, $\mathcal{H}^1(\ell^1(A), X)$ is a Banach space for all commutative $A$-module $X$, and that $\mathcal{H}^1(\ell^1(S), \ell^\infty(S)) = 0$.

**Theorem 2.1.** Let $A = \ell^1(S)$, where $S$ is a semilattice, and let $X$ be a commutative $A$-module. Then $\mathcal{H}^1(\ell^1(A), X)$ is a Banach space.

**Proof.** Let $\phi \in \mathcal{C}^1(A, X)$. We define $\psi \in \mathcal{C}^1(A, X)$ by

$$ \psi(u) = (2u - 1)\phi(u, u). $$

For $\phi' \in \mathcal{C}^2(A, X)$ given by $\phi'(u, v) = \phi(u, v) - \delta \psi(u, v)$, we have $\delta \phi' = \delta \phi \in \mathcal{B}^3(\ell^1(A), X')$. 


We claim that there exists a constant $M$ such that $\|\phi'(u, v)\| \leq M \|\delta\phi\|$ for every $u, v \in S$: as outlined in the introduction, this is equivalent to $H^3(A, X)$ being a Banach space.

Let us prove our claim. We have

$$\phi'(u, u) = \phi(u, u) - 2u\psi(u) + \psi(u)$$
$$= \phi(u, u) - 2u(2u - 1)\phi(u, u) + (2u - 1)\phi(u, u) = 0,$$

as $X$ is a commutative module. Using the 2--coboundary map, for every $a, b, c \in S$ we have

$$\|\delta\phi(a, b, c)\| = \|a\phi'(b, c) - \phi'(ab, c) + \phi'(a, bc) - \phi'(a, b)c\| \leq \|\delta\phi\|. \tag{2.1}$$

Let $u, v \in S$ be such that $uv = v$. Using (2.1) with $u, v, v$ instead of $a, b, c$, respectively, we obtain (using $\phi'(u, u) = 0$)

$$\|u\phi'(u, v)\| \leq \|\delta\phi\|. \tag{2.2}$$

Using (2.1) with $u, v, v$, along with $\phi'(v, v) = 0$ and commutativity of the module actions, we obtain

$$\|v\phi'(u, v)\| \leq \|\phi'(u, v) - v\phi'(u, v)\| \leq \|\delta\phi\|. \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$\|v\phi'(u, v)\| \leq \|u(1 - v)\phi'(u, v)\| + \|u\phi'(u, v)\| \leq 2\|\delta\phi\|.$$ 

Thus for every $u, v \in S$ with $uv = v$ we have

$$\|\phi'(u, v)\| \leq \|\phi'(u, v) - v\phi'(u, v)\| + \|v\phi'(u, v)\| \leq 3\|\delta\phi\|. \tag{2.4}$$

If we now consider any $u, v \in S$, we deduce from (2.4) that

$$\|\phi'(u, uv)\| \leq 3\|\delta\phi\|.$$ 

Using (2.1) with $u, u, v$ we now obtain

$$\|(1 - v)\phi'(u, v)\| = \|\phi'(u, v) - u\phi'(u, v)\| \leq \|\delta\phi\| + \|\phi'(u, uv)\| \leq 4\|\delta\phi\|.$$

A similar argument to the one deployed above (starting before (2.2), applying (2.1) for $uv = u$) yields $\|(1 - v)\phi'(u, v)\| \leq 4\|\delta\phi\|$.

Using (2.1) with $u, v, uv$ gives

$$\|u\phi'(v, uv) + \phi'(u, uv) - \phi'(u, v)uv\| \leq \|\delta\phi\|.$$ 

Thus

$$\|uv\phi'(u, v)\| \leq \|uv\phi'(u, v) - u\phi'(v, uv) - \phi'(u, uv)\| + \|u\phi'(v, uv)\| + \|\phi'(u, uv)\| \leq 7\|\delta\phi\|,$$

and we deduce that

$$\|v\phi'(u, v)\| \leq \|v\phi'(u, v) - uv\phi'(u, v)\| + \|uv\phi'(u, v)\|$$
$$\leq 11\|\delta\phi\|.$$ 

Therefore

$$\|\phi'(u, v)\| \leq \|(1 - v)\phi'(u, v)\| + \|v\phi'(u, v)\|$$
$$\leq 15\|\delta\phi\|$$

which proves our claim, and the proof is complete. \qed
Theorem 2.2. Let $S$ be a semilattice. Then $\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = 0$.

Proof. Let $F$ be a finite subset of $S$ and let $S_F = \{e_i : i \in J\}$ be the finite semigroup generated by $F$, where $J$ is a finite index set. Then $\ell^1(S_F)$ is finite dimensional and it is the image of a finite dimensional group algebra given as follows.

For each $e_i \in S_F$ set $F$ generated by $F$ and it is the image of a finite dimensional group algebra given as follows.

Let $\psi$ be a continuous and surjective homomorphism. This shows that $\psi \circ \delta \psi$ is a bounded linear operator $t$ if $H(t)$ is invertible and the $\delta$ is amenable.

Now define a function $\tilde{\psi}$ such that $\psi_F = \delta \psi_F$ and $\|\psi_F\| \leq 15 \|\phi_F\|$. Now define a function $\tilde{\psi}_F \in C^2(\ell^1(S), \ell^\infty(S))$ by extending $\tilde{\psi}_F$ to be zero for $\tilde{\psi}_F(s_1, s_2)(s_3)$ if any $s_i \notin S_F$ for $i = 1, 2, 3$. $\tilde{\psi}_F$ is a bounded net in $C^2(\ell^1(S), \ell^\infty(S)) \simeq \ell^\infty(S \times S \times S)$ which has a compact ball. Now let $\tilde{\psi}'$ be a limit point of some subnet of the $\tilde{\psi}_F$. Then

$$\tilde{\psi}'(s_1, s_2)(s_3) = \lim \tilde{\psi}_F(s_1, s_2)(s_3).$$

Therefore

$$\delta \tilde{\psi}'(s_1, s_2)(s_3) = \lim \delta \tilde{\psi}_F(s_1, s_2)(s_1)$$
$$= \lim \phi_F(s_1, s_2)(s_3)$$
$$= \phi(s_1, s_2)(s_3).$$

Thus $\delta \tilde{\psi}' = \phi$ and we have

$$\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = \mathcal{B}^3(\ell^1(S), \ell^\infty(S)),$$

which means $\mathcal{H}^3(\ell^1(S), \ell^\infty(S)) = 0$. \hfill $\square$

To show the vanishing of the cohomology group $\mathcal{H}^{n+1}(A, A')$, we construct a bounded linear operator $t^n : C^{n+1}(A, A') \to C^n(A, A')$ ($t^n$ is called the contracting homotopy) such that $\delta^n t^n + t^{n+1} \delta^{n+1}$ is the identity map on $C^{n+1}(A, A')$, because if $\phi \in C^{n+1}(A, A')$ such that $\delta^{n+1} \phi = 0$, then $\delta^n(t^n(\phi)) = \phi$, which means that $\mathcal{H}^{n+1}(A, A')$.

2.3. An alternate proof for Theorem 2.2:

Proof. Let $A = \ell^1(S)$, where $S$ is a semilattice, and let $T \in C^3(A, A')$. We define

$$t^2(T)(u,v) = 2uvT(u,u,uv) + uvT(v,v,vv) - uvT(uv,uv,vv)$$
$$+ uT(v,uv,v) + uT(u,v,v) - uT(uv,uv,v)$$
$$+ 2T(u,u,uv) - T(u,v,uv) - T(u,u,v).$$
We claim that $\delta^1 + t^2 \delta^2 = \text{id}$, where $t^1 : C^2(A, A') \to C^1(A, A')$ is defined by $t^1(\phi)(e) = (2e - 1)\phi(e, e)$. To prove our claim for $\phi \in C^2(A, A')$ we have
\[ t^2(\delta^2)(\phi)(u, v) = 2uv\delta^2(\phi, u, u, v) + uv\delta^2(\phi, v, v, u) - uv\delta^2(\phi, u, v, v) \]
\[ + u\delta^2(\phi, v, v, v) + u\delta^2(\phi, u, v, v) - u\delta^2(\phi, uv, uv, v) \]
\[ + 2\delta^2(\phi, u, u, uv) - \delta^2(\phi, u, v, uv) - \delta^2(\phi, u, uv, v). \]

Using the definition of boundary map $\delta^2$ we obtain the value of all terms on the right-hand side of the above as follows
\[ t^2(\delta^2)(\phi)(u, v) = \phi(u, v) - [u(2v - 1)\phi(v, v) - (2uv - 1)\phi(u, v) + v(2u - 1)\phi(u, u)] \]
\[ = (id - \delta^1 t^1)(\phi)(u, v), \]
which proves our claim, and the proof is complete. \qed

3. Approximately additive functions and the semigroup $\mathbb{N}_a$

**Definition 3.1.** A real-valued function $f$ defined on a subset $X$ of a semigroup $S$ is called 1-additive if
\[ |f(x) + f(y) - f(x + y)| < 1 \] when $x, y, x + y \in X$,
and additive if
\[ |f(x) + f(y) - f(x + y)| = 0 \]
when $x, y, x + y \in X$.

The following proposition will enable us to deduce that the boundary map
\[ \delta : C^1(\ell^1(\mathbb{N}_a), \ell^\infty(\mathbb{N}_a)) \to C^2(\ell^1(\mathbb{N}_a), \ell^\infty(\mathbb{N}_a)) \]
is open onto its range, and hence that $H^2(\ell^1(\mathbb{N}_a), \ell^\infty(\mathbb{N}_a))$ is a Banach space.

**Proposition 3.2.** Let $f$ be a real-valued 1-additive function on $[s, t] = \{n \in \mathbb{N} : s \leq n \leq t\}$. Then there exists a universal constant $K$ and an additive function $g$ on $[s, t]$ such that $\|f - g\|_\infty < K$ where $\|f\|_\infty = \max_{x \in [s, t]} |f(x)|$.

**Proof.** We can assume that $f(t) = 0$ by subtracting the linear function $g(x) = \frac{t}{t} f(t)$ which is additive. The proof will proceed through four cases.

Case 1: $t < 2s$.

In this case, any function is additive as there are no constraints, and we let $g = f$.

Case 2: $t = 2s$.

As $f(2s) = 0$, we have $1 > |f(s) + f(s) - f(2s)| = |2f(s)|$ and thus $|f(s)| < 1/2$. Letting $g(s) = g(2s) = 0$ and $g(x) = f(x)$ for $s < x < 2s$, we have $g$ additive and $\|f - g\|_\infty < 1/2$.

Case 3: $2s < t < 3s$.

Here $t = 2s + u$ with $0 < u < s$. Let $s_1 = \lfloor t/2 \rfloor$, $I_1 = [s, s_1]$, $I_2 = [s_1 + 1, t - s]$, $I_3 = [t - s + 1, 2s - 1]$ and $I_4 = [2s, t]$. (Note that $I_3 = \emptyset$ if $t = 3s - 1$; all intervals are otherwise non-empty.)

The first step is to show that we can assume $f(s) = 0$, $f(2s) < 1$ and $f$ zero on $I_3$. To do so, let us consider the functions $g_1$ and $g_2$ defined by

\[
g_1(x) = \begin{cases} 
  f(x) & \text{if } x \in I_3 \\
  0 & \text{otherwise}
\end{cases}
\]

and
and
\[
g_2(x) = \begin{cases} 
(1 - 2j/u)f(s) & \text{if } x = s + j \text{ for } 0 \leq j \leq u \\
(2 - 2j/u)f(s) & \text{if } x = 2s + j \text{ for } 0 \leq j \leq u \\
0 & \text{otherwise}
\end{cases}
\]

Note that for each \(x\), at most one of these two functions is non-zero.

These two functions are additive. For \(g_1\), this follows from the observation that for all \(x \in I_3\), we have \(s + s > x\) and \(s + x > t\), and thus there are no constraint involving non-zero values of \(g_1\). For \(g_2\), it is an easy check. We can therefore subtract these two functions from \(f\) and the resulting function will still be \(1\)-additive, and will vanish on \(I_3\) and at \(s\).

Therefore, without loss of generality, we can assume that \(f\) is \(1\)-additive, that \(f(s) = 0\) (from which we immediately deduce \(f(2s) < 1\)) and that \(f\) is zero on \(I_3\).

We show that such a function cannot take values of modulus greater or equal to \(5\).

Let \(M = \|f\|_{\infty}\) and let \(x_0\) be the smallest value such that \(|f(x_0)| = M\). Without loss of generality, assume \(f(x_0) = M\). If \(x_0 \in I_1\), then \(|f(2x_0) - 2f(x_0)| < 1\) and therefore \(2M - M < 1\). Thus \(M < 1\) and we are done.

If \(x_0 \not\in I_1\), then for any \(x \in I_1\), we must have \(|f(2x) - 2f(x)| < 1\) and therefore \(|f(x)| < \frac{M}{2} + \frac{1}{2} \) (as \(|f(2x)| \leq M\)). If \(x_0 \in I_2\), then \(t - x_0 \in I_1\), and thus from \(|f(x_0) + f(t - x_0) - f(t)| < 1\) we can deduce that \(M = f(x_0) < |f(t - x_0)| + 1 < \frac{M}{2} + \frac{5}{2}\).

This gives \(M < 3\) and we are done.

If \(x_0 \not\in I_1 \cup I_2\), then for \(x \in I_2\) we have \(t - x \in I_1\) and \(|f(x) + f(t - x)| < 1\). As \(f(t - x) < \frac{M}{2} + \frac{1}{2} \) (from (4)), we obtain \(|f(x)| < \frac{M}{2} + \frac{3}{2}\).

The only non-trivial case remaining is \(x_0 \in I_4\) as \(f\) vanishes on \(I_3\). Then \(x_0 - s \in I_1 \cup I_2 \cup I_3\) and from the estimates already obtained we have \(|f(x_0 - s)| < \frac{M}{2} + \frac{3}{2}\).

Using this estimate, we deduce from \(|f(x_0 - s) + f(s) - f(x_0)| < 1\) that \(M = f(x_0) < |f(x_0 - s)| + 1 < \frac{M}{2} + \frac{5}{2}\), and therefore \(M < 5\).

Case 4: \(t \geq 3a\).

Let \(M = \|f\|_{\infty}\), let \(x_0\) be the smallest value such that \(|f(x_0)| = M\), assuming without loss of generality that \(f(x_0) = M\), let \(I_1\) and \(I_2\) be defined as in Case 3, and let \(I_3 = [t - s + 1, t - 1]\).

If \(x_0\) is in \(I_1\) or \(I_2\), we argue as we did in Case 3 and we get \(M < 1\) or \(M < 3\).

Suppose now that \(x_0 \in I_3\). We have \(|f(s) + f(x_0 - s) - f(x_0)| < 1\). As \(x_0 - s \in I_1 \cup I_2\), we have \(|f(x_0 - s)| < \frac{M}{2} + \frac{3}{2}\) as in Case 3. Note that we also have some control on \(-f(s)\) as \(2s \in I_1 \cup I_2\): we have \(|f(2s)| < \frac{M}{2} + \frac{3}{2}\) and, as \(|2f(s) - f(2s)| < 1\), we get \(|f(s)| < \frac{M}{2} + \frac{3}{2}\). Thus we get \(M < 1 + \frac{M}{2} + \frac{3}{2} + \frac{M}{2} + \frac{3}{2}\) and we obtain \(M < 15\). Hence we have proven the proposition with the constant \(K = 15\). □

Remark 3.3. The proof of the previous proposition is long. If one tries to simplify the proof by extending the function, then the problem is that we cannot extend the definition of a \(1\)-additive function on \([s, t]\) to a \(1\)-additive function on \([s, \infty]\): an easy example is provided by the \(1\)-additive function \(f\) on \([3, 5]\) defined by \(f(3) = f(5) = 10\) and \(f(4) = 0\), which cannot be extended to take a value on \(8\) for instance. Also, we can not in general subtract an additive function \(g\) in such a way that \((f - g)(s) = (f - g)(t) = 0\): this would give an easier argument. Finally, note that the proposition may well hold with a smaller constant but this is not something of concern for us.

Theorem 3.4. With the notation as above, \(H^2(\ell^1(N_a), \ell^\infty(N_a))\) is a Banach space.
Proof. Let $\phi \in C^1(\ell^1(N_n), \ell^\infty(N_n))$ be such that $\|\delta\phi\| < 1$. Using the one-to-one correspondence between $C^0(\ell^1(N_n), \ell^\infty(N_n))$ and bounded functions from the $n$-fold product $N_n \times \cdots \times N_n$ into $\ell^\infty(N_n)$, we write

$$|\delta\phi(x,y)(z)| < 1 \quad \forall x, y, z \in N_n,$$

which is

$$|\phi(y)(x+z) - \phi(x+y)(z) + \phi(x)(y+z)| < 1.$$  

For each $N \geq 3a$, let $f_N : [a, N - a] \to \mathbb{R}$ be given by

$$f_N(x) = \phi(x)(N - x).$$

Then $f_N$ is 1-additive as, for $x, y, x + y \in [a, N - a]$, we have

$$|f_N(x) + f_N(y) - f_N(x + y)| = |\delta\phi(x,y)(N - (x + y))| < 1.$$  

Therefore it follows from Proposition 3.2 that, for each $N \geq 3a$, there exists $g_N : [a, N - a] \to \mathbb{R}$ additive such that $\|f_N - g_N\|_{\infty} < K$ for a fixed constant $K$.

Let $\psi \in C^1(\ell^1(N_n), \ell^\infty(N_n))$ be induced by

$$\psi(x)(y) = \begin{cases} \phi(x)(y) & \text{if } x + y < 3a; \\ g_N(x) & \text{else, where } N = x + y. \end{cases}$$

Then $\delta(\phi - \psi) = \delta(\phi)$ and $\|\delta - \psi\| < K$. The map $\delta$ is therefore open onto its range, which proves the theorem. \hfill $\Box$

4. Clifford semigroup algebra

In this section, we show that $H^2(\ell^1(S), \ell^\infty(S))$ is a Banach space, where $S$ is a Clifford semigroup. We recall that $S$ is a Clifford semigroup if it is an inverse semigroup with each idempotent central, or equivalently, if it is a strong semilattice of groups. So we can write our Clifford semigroup as $S = \cup \{G_e : e \in E\}$ where $E$ is the semilattice of idempotents and each $G_e$ is a group with identity element $e$, and for every $e, e' \in E$, we have $G_e, G_{e'} \subseteq G_{ee'}$ ([2, Section 4.2]).

Remark 4.1. Let $\phi \in C^2(\ell^1(S), \ell^\infty(S))$ be a 2-cocycle. Then by [3, Theorem 2.5] there exists a $\psi \in C^1(\ell^1(S), \ell^\infty(S))$ such that $\phi = \delta\psi$ on $E$. So if we define $\phi' = \phi - \delta\psi$, then $\phi'(e_1, e_2) = 0$ for every $e_1, e_2 \in E$. Thus without loss of generality by replacing $\phi$ by $\phi - \delta\psi$, if it is necessary, we can assume that for any 2-cocycle $\phi$ we have $\phi(e_1, e_2) = 0$, where $e_1, e_2 \in E$.

If $\phi \in C^2(\ell^1(S), \ell^\infty(S))$ is a 2-cocycle, then for every $e \in E$ and $h \in S$ such that $eh = h$, by the 2-cocycle equation $\delta\phi(e, e, h) = 0$, we have $\phi(h, e) = 0$ and similarly $\phi(h, e) = 0$

Lemma 4.2. Let $\phi$ be a 2-cocycle. Then there exists $\psi \in C^1(\ell^1(S), \ell^\infty(S))$ such that $(\phi - \delta\psi)(g, e) = 0$ for every $g \in S$, and $e \in E$ with $eg = g$.

Proof. If $eg = g$, then $g \in G_{ee'}$ for some $e' \in E$ such that $ee' = e'$, (for instance, taking $e' = g^{-1}y$). Using the 2-cocycle equation $\delta\phi(e', g, e) = 0$, we obtain

$$e'\phi(g, e) - \phi(g, e) + \phi(e', g) - \phi(e', g)e = 0.$$  

Since $eg = g$ and $ee' = e'$ we have

$$e'\phi(g, e) = e'e\phi(g, e) = 0,$$
thus $\phi(g, e) = (1 - e)\phi(e', g)$ whenever $eg = g$. Set $\psi(g) = -\phi(e, g)$ for every $g \in G_e$. Then for every $g \in G_e$,

$$(\phi - \delta\psi)(g, e) = \phi(g, e) - g\psi(e) + \psi(g) - \psi(e)(g)e$$

$$= \phi(g, e) - \phi(e', g) + \phi(e', g)e$$

$$= \phi(g, e) - (1 - e)\phi(e', g) = 0,$$

whenever $eg = g$.

\[\Box\]

Remark 4.3. By the previous Lemma, by replacing $\phi$ by $\phi - \delta\psi$, if it is necessary, we can assume that $\phi(g, e) = 0$, whenever $ge = g$. Applying the 2-cocycle equation $\delta\phi(e, g, h)(k) = 0$ for $e \in E$, $g, h, k \in S$ with $ge = e$, we obtain (using $\phi(g, e) = 0$)

$$\phi(g, h)(k) = \phi(g, h)(ek).$$

Similarly $\phi(g, h)(k) = \phi(g, h)(ek)$, whenever $he = e$.

Lemma 4.4. Let $\phi$ be a 2-cocycle and let $\psi$ be defined by $\psi(g)(h) = \phi(g, e')(h)$, for $g \in G_{e_1}$ and $h \in G_{e_2}$, where $e' = e_1e_2$. Then

$$(\phi - \delta\psi)(g, e)(h) = 0$$

for every $g, h \in S$ and $e \in E$.

Proof. For every $g, h \in S$ and $e \in E$, we have

$$\delta\psi(g, e)(h) = \psi(e)(hg) - \psi(ge)(h) + \psi(g)(eh)$$

$$= \phi(e, e')(hg) - \phi(ge, e')(h) + \phi(g, e')(eh)$$

$$= \phi(e, e')(hg) - \phi(ge, e')(h) + \phi(g, e')(h)$$

and, since $\delta\phi(g, e, e')(h) = 0$, we get

$$= \phi(g, e)(e'eh) = \phi(g, e)(e_1eh)$$

$$= \phi(g, e)(eh) = \phi(g, e)(h),$$

where we have used several times Remark 4.3.

\[\Box\]

By Lemma 4.4 without less of generality we can assume that for any 2-cocycle $\phi$ we have $\phi(g, e)(h) = 0$, for every $g, h \in S$ and $e \in E$.

Lemma 4.5. For every 2-cocycle $\phi$ and for every $g \in G_{e_1}$, $h \in G_{e_2}$, $k \in G_{e_3}$ and $e = e_1e_2e_3$, we have

$$\phi(ge, he)(ke) = \phi(g, h)(ke).$$

Proof. By the 2-cocycle equation $\delta\phi(e, g, h)(ke) = 0$, we have

$$g\phi(e, he)(ke) - \phi(ge, he)(ke) + \phi(g, he)(ke) - \phi(g, h)(ke) = 0.$$

By Lemma 4.4, the first and the last terms of the above equation are zero, and therefore

$$\phi(ge, he)(ke) = \phi(g, h)(ke).$$

The 2-cocycle equation $\delta\phi(g, he, e)(ke) = 0$ gives

$$\phi(ge, he)(ke) = \phi(g, h)(ke).$$

Finally by Remark 4.3, since $ge = g$ and $he = h$, we have

$$\phi(g, h)(ke) = \phi(g, h)(ke_1e_2) = \phi(g, h)(k).$$

\[\Box\]
Theorem 4.6. Let $S$ be a Clifford semigroup and let $\psi \in C^1(\ell^1(S), \ell^\infty(S))$ with $\|\delta \psi\| < 1$. Then there exists a constant $M$ and $\hat{\psi} \in C^1(\ell^1(S), \ell^\infty(S))$ such that $\|\hat{\psi}\| < M$ and $\delta \hat{\psi} = \delta \psi$.

Proof. For every $g \in S$ and $e \in E$, we have

(4.1) \[ |\delta \psi(e, g)(g)| = |2\psi(e)(ge) - \psi(e)(g)| < 1. \]

For $ge$ instead of $g$ in (4.1), we obtain $|\psi(e)(ge)| < 1$. Thus

(4.2) \[ |\psi(e)(g)| < 3. \]

For $g \in G_{e_1}$, $h \in G_{e_2}$ and $e' = e_1e_2$, we have

\[ |\delta \psi(e_1, g)(h)| = |\psi(h)(he_1) - \psi(g)(h) + \psi(e_1)(gh)| < 1 \]
\[ \text{and} \]
\[ |\delta \psi(e_2, g)(he_1)| = |\psi(g)(he') - \psi(ge_2)(he_1) + \psi(e_2)(gte_1)| < 1. \]

From (4.2), we have, using respectively $he' = he_1$ and $ge' = ge'$,

\[ |\psi(g)(h) - \psi(ge')(he')| < 4 \]
and therefore

(4.3) \[ |\psi(g)(h) - \psi(ge')(he')| < 8. \]

Now for $g \in G_{e_1}$, $h \in G_{e_2}$ and $e' = e_1e_2$, we define $\psi_0(g)(h) = \psi(e')g(e'h)$. Then by (4.3), we have

$\|\psi - \psi_0\| < 8.$

For every $e \in E$ let us define $\psi_e \in C^1(\ell^1(G_e), \ell^\infty(G_e))$ by $\psi_e(g)(h) = \psi_0(g)(h)$. It is clear that $\|\delta \psi_e\| \leq \|\delta \psi_0\| < 1$, thus by Theorem 1.1 there exists a $\psi'_e \in C^1(\ell^1(G_e), \ell^\infty(G_e))$ such that $\|\psi'_e\| \leq 2$ and $\delta \psi'_e = \delta \psi_e$.

Let $\psi': \ell^1(S) \rightarrow \ell^\infty(S)$ be given by

$\psi'(g)(h) = \psi_e'(ge')(he'),$

where $g \in G_{e_1}$, $h \in G_{e_2}$ and $e' = e_1e_2$. Now we need to show that $\delta \psi' = \delta \psi_0$. By Lemma 4.5 for $g \in G_{e_1}$, $h \in G_{e_2}$, $k \in G_{e_3}$ and $e' = e_1e_2e_3$, we have

$\delta \psi'(g, h)(k) = \delta \psi'_e(ge', he')(ke')$
$= \delta \psi_0(ge', he')(ke')$
$= \delta \psi_0(g, h)(k).$

Now define $\hat{\psi}$ by $\hat{\psi} = \psi - \psi_0 + \psi'$. We have

$\delta \psi = \delta(\psi - \psi_0) + \delta \psi' = \delta(\psi - \psi_0) + \delta \psi'$
and

$\|\hat{\psi}\| = \|\psi - \psi_0 + \psi'\| \leq \|\psi - \psi_0\| + \|\psi'\| \leq 10.$

Corollary 4.7. Let $S$ be a Clifford semigroup. Then $\mathcal{H}^2(\ell^1(S), \ell^\infty(S))$ is a Banach space.
5. Rees semigroup algebra

In this section we calculate the second simplicial cohomology group for the class of all completely 0-simple semigroup. Up to isomorphism, these semigroups are just the Rees semigroups \[7\]. Let \(G\) be a group, \(I\) and \(\Lambda\) be index sets, and \(G^0 = G \cup \{0\}\) be the group with zero arising from \(G\) by adjunction of a zero element. Let \(P = (p_{\lambda j})\) be a sandwich matrix over \(G^0\), each row and column of \(P\) contains a nonzero entry. The associated Rees semigroup is defined by \(S = I \times G \times \Lambda \cup \{0\}\), where \(0\) acts as the zero element of \(S\) and

\[(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu),\]

if \(p_{\lambda j} \neq 0\) and \(0\) otherwise.

**Theorem 5.1.** Let \(S\) be a Rees semigroup. Then the cyclic cohomology \(HC^2(\ell^1(S))\) is a Banach space.

**Proof.** Choose \(k \in I, \alpha \in \Lambda\) with \(p_{\alpha k} \neq 0\). Let \(g = p_{\alpha k}^{-1}\). Let \(\phi \in CC^1(\ell^1(S))\), such that \(|\delta \phi| < 1\), and take \(\phi_\alpha(g)(h) = \phi(k, qg, \alpha)(k, qh, \alpha)\). Thus \(\phi_\alpha \in C^1(\ell^1(G), \ell^\infty(G))\) and \(\|\delta \phi_\alpha\| < 1\) so by Theorem 1.1 there exist a \(\psi_\alpha \in \ell^\infty(G)\) such that

\[
|\delta \psi_\alpha(g) + \phi_\alpha(g)| < 2.
\]

Define \(\psi \in \ell^\infty(S)\) by \(\psi(0) = 0\) and

\[
\psi(s) = \psi_\alpha(p_{\lambda j}g),
\]

where \(s = (i, g, \lambda)\). For every \(s = (i, g, \lambda)\), set \(u = (k, q, \lambda)\) and \(x = (i, g, \alpha)\) and define

\[
\tilde{\psi}(s) = \psi(s) + \phi(u)(x),
\]

and \(\tilde{\psi}(0) = 0\).

We claim that there exists a constant \(M > 0\) such that \(|\delta \tilde{\psi}(s)(t) + \phi(s)(t)| < M\). To see this we use the present notation to show the real part of all functions. For \(s = (i, g, \lambda)\) and \(t = (j, h, \mu)\) suppose first that \(st \neq 0\), that is \(p_{\lambda j} \neq 0\). (By the symmetry of the cyclic condition we may suppose that \(ts \neq 0\).)

Consider \(u = (k, q, \lambda)\), \(x = (i, g, \alpha)\), then

\[
xu = (i, g, \alpha)(k, q, \lambda) = (i, gp_{\alpha k}q, \lambda) = s,
\]

and

\[
ux = (k, q, \lambda)(j, h, \mu) = (k, qp_{\lambda j}h, \mu) = (k, q, \mu) = yv,
\]

where \(y = (k, qp_{\lambda j}h, \alpha)\) and \(v = (k, q, \mu)\). So since \(\phi\) is cyclic and real, we have

\[
\phi(s)(t) = \phi(xu)(t) \leq \phi(u)(tx) + \phi(x)(ut) + \|\delta \phi\|
\]

\[
= \phi(u)(tx) - \phi(ut)(x) + \|\delta \phi\|
\]

\[
= \phi(u)(tx) - \phi(yv)(x) + \|\delta \phi\|
\]

\[
\leq \phi(u)(tx) - \phi(y)(vx) - \phi(v)(xy) + 2\|\delta \phi\|
\]

\[
= \phi(u)(tx) + \phi(v)(xy) - \phi(v)(xy) + 2\|\delta \phi\|.
\]

\[
(5.3)
\]
But from (5.1)
\[
\phi(xy)(y) - \phi(k, q\mu_g, \alpha)(k, q\lambda_j, \alpha) = 0
\]
(5.4)
\[
\begin{align*}
\psi_s & \equiv \psi_j(p_{\mu_i}g) + \psi_p(p_{\lambda_j}h) + 2 \|\delta\phi\| \\
& = -\psi_p(p_{\mu_i}g p_{\lambda_j}h) + \psi_p(p_{\lambda_j}h p_{\mu_i}g) + 2 \|\delta\phi\| \\
& = -\psi(i, g p_{\lambda_j}h, \mu) + \psi(j, h p_{\mu_i}g, \lambda) + 2 \|\delta\phi\| \\
& = -\psi(st) + \psi(ts) + 2 \|\delta\phi\|.
\end{align*}
\]
Using (5.2) for \( ts = (j, h p_{\mu_i}g, \lambda) \), set \( u = (k, q, \lambda) \) and \( tx = (j, h p_{\mu_i}g, \alpha) \) we have
\[
\tilde{\psi}(ts) = \psi(ts) + \phi(u)(tx),
\]
and similarly for \( st = (i, g p_{\lambda_j}h, \mu) \), set \( v = (k, q, \mu) \) and \( xy = (i, g p_{\lambda_j}h, \alpha) \) we have
\[
\tilde{\psi}(st) = \psi(st) + \phi(v)(xy).
\]
Thus if we replace (5.4) in (5.3) we obtain
\[
\begin{align*}
\phi(s)(t) & \leq \phi(u)(tx) + \phi(xy)(y) - \phi(v)(xy) + 2 \|\delta\phi\| \\
& \leq [\phi(u)(tx) + \psi(ts)] - [\phi(v)(xy) + \psi(st)] + 2 \|\delta\phi\| \\
& = \tilde{\psi}(ts) - \tilde{\psi}(st) + 4 \|\delta\phi\|.
\end{align*}
\]
A similar result holds for the imaginary part of all functions. So we have proved that \( \|\delta\tilde{\psi}(s)(t) + \phi(s)(t)\| < 8 \). Now if we define \( \tilde{\psi}(s)(t) = \delta\tilde{\psi}(s)(t) + \phi(s)(t) \). Then \( \delta\tilde{\psi} = \delta\phi \) and \( \|\tilde{\psi}\| \leq 8 \).

In the case where \( st = 0 \), that is, \( p_{\lambda_j} = 0 \), if \( p_{\mu_i} = 0 \), then by definition \( \tilde{\psi}(st) = \tilde{\psi}(ts) = 0 \) and clearly the results holds. If \( p_{\mu_i} \neq 0 \), then \( ut = 0 \) and \( tx = (j, h p_{\mu_i}g, \alpha) \) by the same calculation as we show in the previous case we have \( \|\tilde{\psi}(st) - \tilde{\psi}(0) + \phi(s)(t)\| \leq 8 \), and the proof is complete. \( \Box \)

**Definition 5.2.** Let \( \mathcal{A} \) be a Banach algebra. Then the 2-cocycle \( \phi \in C^2(\mathcal{A}, \mathcal{A}') \) is called equivalent to cyclic cocycle if there exists \( \psi \in C^1(\mathcal{A}, \mathcal{A}') \) such that \( \phi + \delta\psi \) is a cyclic 2-cocycle.

Let \( S \) be a Rees semigroup, to which we adjoin a unit element 1 to give \( S^1 = S \cup \{1\} \). Since \( \ell^1(S^1) \) is unital, the Connes-Tzygan exact sequence exists, that is,
\[
\mathcal{H}C^0(\ell^1(S^1)) \xrightarrow{\theta} \mathcal{H}C^2(\ell^1(S^1)) \xrightarrow{\tau} \mathcal{H}C^2(\ell^1(S^1), C_0(S^1)) \xrightarrow{\ell} \mathcal{H}C^1(\ell^1(S^1)) \rightarrow \cdots,
\]
where \( \mathcal{H}C^0(\ell^1(S^1)) \) is the Banach space of traces, \( \mathcal{H}C^1(\ell^1(S^1)) = 0 \) (11) and, by the previous Theorem, \( \mathcal{H}C^2(\ell^1(S^1)) \) is a Banach space.

We note that for every \( \tau \in \mathcal{H}C^0(\ell^1(S^1)) \) and \( a, b, c \in S \), \( \theta(\tau)(a, b)(c) = \tau(ab) \); \( \iota \) is an inclusion map; and for \( \phi \in Z^2(\ell^1(S^1), C_0(S^1)) \), we have
\[
\eta(\phi)(a)(b) = \phi(a, b)(1) - \phi(b, a)(1),
\]
and as \( \mathcal{H}C^1(\ell^1(S^1)) = 0 \), there exists \( f \in C^\infty(S^1) \) such that
\[
\eta(\phi)(a)(b) = \delta f(a)(b) = f(ab) - f(ba).
\]

**Lemma 5.3.** If \( \phi \in Z^2(\ell^1(S^1), C_0(S^1)) \), then there exists a \( \psi \in C^1(\ell^1(S^1), C_0(S^1)) \) such that
\[
(\phi + \delta\psi)(a, b)(1) = (\phi + \delta\psi)(b, a)(1).
\]
Proof. If we let \( \psi_1(a)(b) = -\phi(a, b)(1) \) and \( \phi_1(a, b)(c) = \phi(a, b)(c) + \delta\psi_1(a, b)(c) \), then \( \phi_1(1, b)(1) = 0 \). Since \( \mathcal{H}^1(\ell^1(S^1)) = 0 \) and \( \phi_1 \in \mathcal{Z}^2(\ell^1(S^1), \ell^\infty(S^1)) \), then there exists a \( f \in \ell^\infty(S^1) \) such that \( \eta(\phi_1)(a)(b) = \delta f(a)(b) \), that is,

\[
\phi_1(a, b)(1) - \phi_1(b, a)(1) = f(ab) - f(ba).
\]

Now if we define \( \psi_2(b)(a) = -\phi(a, b)(1) + b \cdot f(a) \), then \( \psi_2(ab)(1) = -\phi_1(ab)(1) + f(ab) = f(ab) \) and similarly \( \psi_2(ba)(1) = f(ba) \). Thus

\[
\phi_1(a, b)(1) - \psi_2(ab)(1) = \phi_1(a, b)(1) - f(ab)
\]

(5.6)

\[
= \phi_1(b, a)(1) - f(ba)
\]

\[
= \phi_1(b, a)(1) - \psi_2(ba)(1).
\]

If we add \( \psi_2(a)(b) + \psi_2(b)(a) \) on both side of (5.6) we obtain

\[
(\phi_1 + \delta \psi_2)(a, b)(1) = (\phi_1 + \delta \psi)(b, a)(1),
\]

that is,

\[
\phi(a, b)(1) + \delta(\psi_1 + \psi_2)(a, b)(1) = \phi(b, a)(1) + \delta(\psi_1 + \psi_2)(b, a)(1).
\]

Hence we have proven the lemma with \( \psi = \psi_1 + \psi_2 \). \( \square \)

By Lemma 5.3 without loss of generality we can assume that all 2-cocycles \( \phi \) satisfy

\[
\phi(a, b)(1) = \phi(b, a)(1).
\]

Lemma 5.4. Let \( S \) be a Rees semigroup. Then every 2-cocycle

\[
\phi \in \mathcal{Z}^2(\ell^1(S^1), \ell^\infty(S^1))
\]

is equivalent to cyclic.

Proof. Let \( \phi \in \mathcal{Z}^2(\ell^1(S^1), \ell^\infty(S^1)) \). If we define \( \psi(a)(b) = -\frac{1}{2} \phi(b, a)(1) \), then as we showed in the proof of Lemma 5.3 we have \( \psi(ab)(1) = -\frac{1}{2} \phi(1, ab)(1) = 0 \). Thus by Lemma 5.3

\[
(\phi + \delta \psi)(a, b)(1) = \phi(a, b)(1) + \psi(a)(b) - \psi(ab)(1) + \psi(b)(a)
\]

\[
= \phi(a, b)(1) - \frac{1}{2} \phi(b, a)(1) - \frac{1}{2} \phi(a, b)(1) = 0.
\]

Now if we define \( \tilde{\phi} = \phi + \delta \psi \), then

\[
\tilde{\phi}(b, c)(a) = \tilde{\phi}(ab, c)(1) = \tilde{\phi}(a, bc)(1) - \tilde{\phi}(a)(b)(c) = 0
\]

since \( \tilde{\phi}(ab, c)(1) = \tilde{\phi}(a, bc)(1) = 0 \), then

\[
\tilde{\phi}(b, c)(a) = \tilde{\phi}(a, b)(c).
\]

Thus \( \phi \) is equivalent to cyclic. \( \square \)

Theorem 5.5. Let \( S \) be a Rees semigroup. Then \( \mathcal{H}^2(\ell^1(S^1), \ell^\infty(S^1)) \) is a Banach space.

Proof. Suppose \( \phi \in \mathcal{Z}^2(\ell^1(S^1), \ell^\infty(S^1)) \) with \( \|\delta \phi\| < 1 \). By Lemma 5.4 there exists a \( \psi \in \mathcal{C}^1(\ell^1(S^1), \ell^\infty(S^1)) \) such that \( \phi + \delta \psi \) is cyclic 2-cocycle and \( \|\delta \phi\| = \|\delta(\phi + \delta \psi)\| < 1 \), so by the proof of Theorem 5.1 there exist a \( \tilde{\phi} \) and \( K > 0 \) such that \( \delta \tilde{\phi} = \delta(\phi + \delta \psi) = \delta \phi \) and \( \|\tilde{\phi}\| < K \). The proof is complete. \( \square \)
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References


DÉPARTEMENT DE MATHEMATIQUES ET DE STATISTIQUE, UNIVERSITÉ LAVAL, CIHÉR UNIVERSE-TAIRE, QUÉBEC, CANADA G1K 7P4
E-mail address: Frederic.Gourdeau@mat.ulaval.ca

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, AMIRKABIR UNIVERSITY OF TECHNOLOGY, 424 HAFEZ AVENUE, TEHRAN 15914, IRAN
E-mail address: arshabbas@aut.ac.ir

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEWCASTLE, NEWCASTLE UPON TYNE, NE1 7RU, ENGLAND
E-mail address: Michael.White@ncl.ac.uk