Chapter 6

Multivariate extremes

In this section we consider the problems we face if we wish to model the extremal behaviour of two or more (dependent) processes **simultaneously**.

There are several reasons why we may wish to do this:

- to model the extreme behaviour of a **particular variable** over **several nearby locations** (e.g. rainfall over a network of sites – simultaneous flooding at several locations could cause devastation);

- to model the joint extremes of two or more **different variables** at a **particular location** (e.g. wind and rain at a site – the combined effects of wind and rain during a hurricane can result in extreme storm surge);

- to model the joint behaviour of extremes which occur as **consecutive observations** in a time–series (e.g. consecutive hourly maximum wind gusts during a storm).

All of these problems suggest fitting an appropriate **limiting multivariate distribution** to the relevant data.

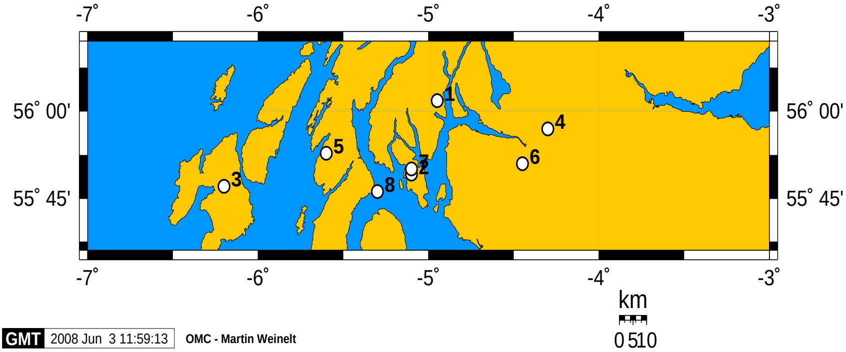
However, as we shall see, the derivation of such a multivariate distribution is not as easy as we might hope.

The analogy with the Normal distribution as a model for means breaks down as we move into *n* dimensions!

It is not even clear what the ‘ **relevant data**’ should be!

Most of the increased complexity is apparent in the move from 1 to 2 dimensions, so we will focus largely on **bivariate problems**.

Suppose we want to study the joint extremes of daily rainfall accumulations at the network of 8 sites is southwest Scotland shown in Figure 6.1.



6.2.1 Example: Rainfall at 8 locations in Scotland

Such issues are of great interest, especially currently – given the **severe flooding** experienced in the U.K.

Suppose we have sequences of daily total rainfall at each location.

There is likely to be **strong inter–site dependence** extremes, in the sense that days with heavy rain are liable to occur simultaneously across locations.

The raw multivariate observations are 8–dimensional vectors of the daily rainfall over the eight sites.

6.2.1 Example: Rainfall at 8 locations in Scotland

Now suppose we wish to take a block–maxima approach, with 'blocks' being years.

For any given year, the 8–dimensional vector of annual maxima is unlikely to be one of the raw multivariate observations.

Let's simplify to the bivariate case: Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be i.i.d. vectors with distribution function $F(x, y)$.

Now consider the **componentwise block maxima**

$$M_{x,n} = \max_{i=1,\ldots,n}\{X_i\} \qquad \text{and} \qquad M_{y,n} = \max_{i=1,\ldots,n}\{Y_i\}.$$

We define the **vector of componentwise maxima** to be

$$\mathbf{M_n} = (M_{x,n}, M_{y,n}).$$

$\mathbf{M_n}$ is not necessarily one of the original observations (X_i, Y_i). Nevertheless, we are interested in the limiting behaviour of $\mathbf{M_n}$ as $n \to \infty$.

The first point to note is that **standard univariate extreme value results** apply in each margin.

When considering the dependence, this allows us to make a simplifying assumption.

6.2.1 Example: Rainfall at 8 locations in Scotland

We assume that the X_i and Y_i variables have a known marginal distribution. It is convenient to assume a unit Fréchet distribution (see Chapter 2), which has CDF

$$F(z) = \exp(-1/z), \qquad z > 0.$$

This gives rise to a very simple normalisation of maxima:

$$\Pr(X_i < x) = \Pr(M_{x,n}/n < x) = \exp(-1/x), \qquad x > 0,$$

(and similarly for Y_i). So if we consider the re–scaled vector

$$\mathbf{M_n^*} = \left(\max_{i=1,\ldots,n} \{X_i\}/n, \max_{i=1,\ldots,n} \{Y_i\}/n \right),$$

the margins are unit Fréchet for all n, and hence we can characterise the limiting joint behaviour of $\mathbf{M_n^*}$ without having to worry about the margins.

6.2.1 Example: Rainfall at 8 locations in Scotland

Unfortunately no limiting parametric family exists (for bivariate extremes, or multivariate extremes in general)!

6.2.2 Limiting distributions for bivariate extremes

Let $\mathbf{M}_\mathbf{n}^* = (M_{x,n}^*, M_{y,n}^*)$ be the normalised maxima as above, where the (X_i, Y_i) are i.i.d. with standard Fréchet marginal distributions.

Then if

$$\Pr(M_{x,n}^*, M_{y,n}^*) \to G(x,y),$$

where G is non–degenerate, then G has the form

$$G\left(x,y\right) = \exp\left\{-V\left(x,y\right)\right\}; \quad x > 0, \;\; y > 0 \tag{6.1}$$

where:

$$V\left(x,y\right) = 2\int_0^1 \max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right) dH\left(\omega\right) \tag{6.2}$$

and H is a distribution function on $[0,1]$ satisfying the mean constraint:

$$\int_0^1 \omega \, dH\left(\omega\right) = 0.5. \tag{6.3}$$

Hence the class of bivariate extreme value distributions is in one-to-one correspondence with distribution functions H satisfying the constraint (6.3).

If H is differentiable with density h, then (6.1) becomes

$$V(x, y) = 2\int_0^1 \max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right) h(\omega) d\omega.$$

However some simple models arise when H is not differentiable. For example, if H places mass 0.5 on each of $\omega = 0$ and $\omega = 1$, then we get

$$G(x, y) = \exp\{-(x^{-1} + y^{-1})\}, \qquad x > 0, y > 0,$$

corresponding to independent x and y.

Since the GEV provides the complete class of **marginal** limit distributions, then the complete class of **bivariate** extreme value distributions is obtained as follows.

If we suppose X and Y are GEV with parameters (μ_x, σ_x, ξ_x) and (μ_y, σ_y, ξ_y) respectively, then the transformations

$$\tilde{x} = \left[1 + \xi_x\left(\frac{x - \mu_x}{\sigma_x}\right)\right]^{1/\xi_x} \quad \text{and} \quad \tilde{y} = \left[1 + \xi_y\left(\frac{y - \mu_y}{\sigma_y}\right)\right]^{1/\xi_y}$$

obtain unit Fréchet margins.

Hence

$$G(x, y) = \exp\{-V(\tilde{x}, \tilde{y})\}$$

is a bivariate extreme value distribution with the appropriate margins for valid $V(.)$.

6.3.1 Example: wave–surge data at Newlyn, Cornwall

Here, we choose a different type of example of dependence to the rainfall problem considered in Section 6.2.1.

Specifically, we consider **two** variables recorded concurrently at the **same site**.

A series of 3–hourly measurements on sea–surge were obtained from Newlyn, southwest England, giving, at each time point,

(i) measurements of the **wave height** (in metres)

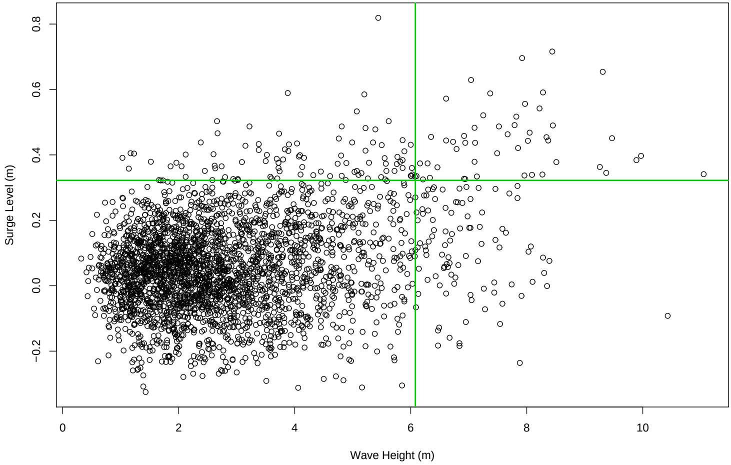
(ii) measurements of the **surge height** (in metres)

Figure 6.2 shows these two variables plotted against each other.

This plot suggests a tendency for extremes of one variable to coincide with extremes of the other.

This dependence could be important – the impact of an event that is **simultaneously extreme** in both variables is likely to be much greater than if extremes of either component occurred in isolation.

6.3.1 Example: wave–surge data at Newlyn, Cornwall



6.3.2 Componentwise maxima to threshold excesses

Now we want to define our bivariate extremes as those observations which exceed a threshold in one or other margin.

For our bivariate observation (X, Y), let's focus on X.

We have already seen that the distribution function for the exceedances of a threshold u by a variable X, conditional on $X > u$ for large enough u, is given by:

$$H(x) = 1 - \lambda_{u_x} \left\{ 1 + \frac{\xi_x (x - u_x)}{\sigma_x} \right\}^{-1/\xi_x}$$

defined on $\{x - u_x : x - u_x > 0 \text{ and } (1 + \xi_x (x - u_x) / \sigma_x) > 0\}$, where $\xi_x \neq 0$, $\sigma_x > 0$, and $\lambda_{u_x} = \Pr(X > u_x)$.

Now we can obtain a unit Fréchet margin with the **transformation**:

$$\tilde{x} = -\left(\log\left\{1 - \lambda_{u_x}\left[1 + \frac{\xi_x\left(x - u_x\right)}{\sigma_x}\right]^{-1/\xi_x}\right\}\right)^{-1}. \qquad (6.4)$$

If we apply the analogous transformation in the Y margin, we obtain

$$G\left(x, y\right) = \exp\left\{-V\left(\tilde{x}, \tilde{y}\right)\right\}; \quad x > u_x, \ \ y > u_y,$$

where $V(x, y)$ is as defined in Equation (6.1), again satisfying the mean constraint in Equation (6.3).

In practice, modelling usually involves identifying a parametric sub–family with appropriate flexibility to handle the structure inherent in the data.

Models can be fitted, e.g. by maximum–likelihood estimation, either:

- **two steps** (marginal components followed by dependence function), or
- or in a **single sweep**

All of these procedures, including the choice of models, are handled in a very similar way when dealing with either bivariate componentwise maxima or bivariate threshold exceedances.

6.4.1 Modelling the dependence structure

The class of bivariate extreme value models contains many families of distributions which can be used to model the dependence structure in the data.

The dependence structure must satisfy the conditions on $H(\omega)$.

Possible choices are:

- **Logistic Model** — symmetric
- Negative Logistic Model
- **Bilogistic Model** — asymmetric
- Dirichlet Model

Here we will focus on the logistic model and the bilogistic model as two commonly used but contrasting choices.

6.4.1 Modelling the dependence structure

Symmetric dependence

- X depends on Y to exactly the same degree that Y depends on X
- They both have the same **influence** over each other

Asymmetric dependence

- X has greater influene over Y than Y has over X (or *vice–versa*)
- Example: Wind speeds at two nearby locations in the U.K.

Here, for $V(x, y)$ in Equation (6.1), we have

$$\left(x^{-1/\alpha} + y^{-1/\alpha}\right)^{\alpha},$$

where $x > 0$, $y > 0$ and $\alpha \in (0, 1)$, giving

$$G(x, y) = \exp\left\{-\left(x^{-1/\alpha} + y^{-1/\alpha}\right)^{\alpha}\right\}.$$

- $\alpha \to 1$ corresponds to independent variables.
- $\alpha \to 0$ corresponds to perfectly dependent variables.
- This model is symmetric — the variables are *exchangeable*.

2. The Bilogistic model

Now we have the following form for $V(x, y)$:

$$-x\gamma^{1-\alpha} - y\,(1-\gamma)^{1-\beta},$$

where $0 < \alpha < 1$, $0 < \beta < 1$ and $\gamma = \gamma\,(x, y; \alpha, \beta)$ is the solution of:

$$(1-\alpha)\,x\,(1-\gamma)^{\beta} = (1-\beta)\,y\gamma^{\alpha}$$

- Independence is obtained when $\alpha = \beta \to 1$ and when one of α or β is fixed and the other approaches 1.
- When $\alpha = \beta$ the model reduces to the logistic model.
- The value of $\alpha - \beta$ determines the extent of **asymmetry** in the dependence structure.

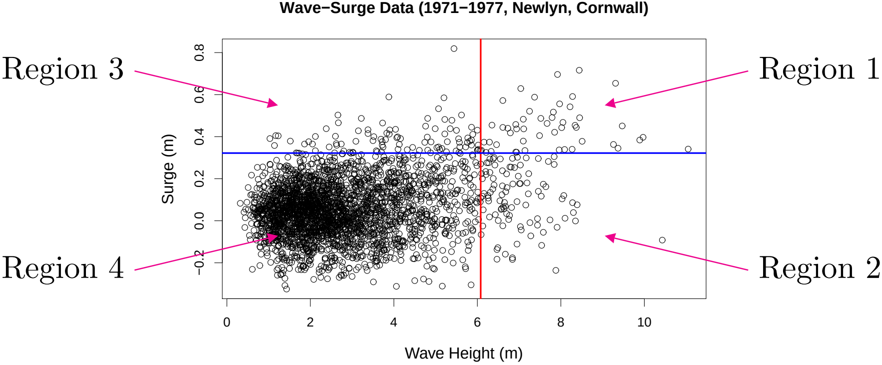
6.4.2 Likelihood calculations

After transformation to unit Fréchet margins, we can obtain the probability density function of the chosen dependence model by **differentiation** of Equation (6.1) to give $g(x, y)$.

From this, the **likelihood** can be formed (and maximised) in the usual way.

6.4.2 Likelihood calculations

However, inference for the bivariate threshold excess setup is complicated by the fact that a bivariate pair may exceed a specified threshold in just one of its components:



6.4.2 Likelihood calculations

We obtain contributions to the likelihood function for a pair of observations in the following way, where θ represents the parameter(s) in our dependence model:

$$g(x,y;\theta) = \begin{cases} \left.\dfrac{\partial^2 G}{\partial x \partial y}\right|_{(\tilde{x},\tilde{y})} & \text{if } (x,y) \in \text{Region 1} \\ \left.\dfrac{\partial G}{\partial x}\right|_{(\tilde{x},\tilde{u}_y)} & \text{if } (x,y) \in \text{Region 2} \\ \left.\dfrac{\partial G}{\partial y}\right|_{(\tilde{u}_x,\tilde{y})} & \text{if } (x,y) \in \text{Region 3} \\ G(\tilde{u}_x,\tilde{u}_y) & \text{if } (x,y) \in \text{Region 4} \end{cases}$$

Then, we have

$$L(\theta; x, y) = \prod_{i=1}^{n} g(\tilde{x}_i, \tilde{y}_i).$$

Example 6.1: Wave–surge analysis at Newlyn

Consider the wave–surge data from Newlyn, Cornwall, shown in Figures 6.2 and 6.3.

Flood defences in Newlyn have been designed to withstand a sea swell resulting from, at most, a **wave height** of $x = 9$metres combined with a **surge height** of $y = 0.7$metres.

A threshold–based approach to modelling is to be used for the wave–surge data shown in Figures 6.2 and 6.3.

Mean residual life plots suggest marginal thresholds of $u_x = 6.1$metres and $u_y = 0.32$metres (as shown in Figure 6.2) for identifying wave height and surge height as extreme.

Example 6.1: Wave–surge analysis at Newlyn

(a) Assuming extreme wave heights occur **independently** of extreme surge heights, find the probability that the flood defence system in Newlyn will be overwhelmed.

Example 6.1: Solution to part (a)

Assuming independence, we have

Pr(flood defence fails)

$$
\begin{aligned}
&= \Pr(X > 9) \times \Pr(Y > 0.7) \\
&= \hat{\lambda}_{u_x}\left[1+\hat{\xi}_x\left(\frac{9-u_x}{\hat{\sigma}_x}\right)\right]_+^{-1/\hat{\xi}_x} \times \hat{\lambda}_{u_y}\left[1+\hat{\xi}_y\left(\frac{0.7-u_y}{\hat{\sigma}_y}\right)\right]_+^{-1/\hat{\xi}_y} \\
&= 0.049\left[1-0.188\left(\frac{9-6.1}{1.334}\right)\right]_+^{1/0.188} \\
&\quad \times 0.051\left[1-0.041\left(\frac{0.7-0.32}{0.093}\right)\right]_+^{1/0.041} \\
&= 0.002995 \times 0.000583 = 0.00000175.
\end{aligned}
$$

(b) Now assume there is **extremal dependence** between wave height and surge height.

(i) Assuming the logistic model for this dependence, obtain the likelihood contributions to $L(\alpha; x_i, y_i)$ if (1) both $x > u_x$ and $y > u_y$; (2) only $x > u_x$; (3) only $y > u_y$; (4) neither x nor y exceed their marginal thresholds.

Example 6.1: Solution to (b)(i)

For the logistic model, we have

$$G(x, y) = \exp\left\{-\left(x^{-1/\alpha} + y^{-1/\alpha}\right)^{\alpha}\right\}.$$

Region 1:

If $x > u_x$ and $y > u_y$, then we are in **Region 1** of Figure 6.3, and so

$$g(x, y; \alpha) = \frac{\partial^2 G}{\partial x \partial y}.$$

Differentiating w.r.t. x gives

$$\begin{aligned}\frac{\partial G}{\partial x} &= \exp\left\{-\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha}\right\} \\ &\quad \times -\alpha\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha-1} \times (-1/\alpha)x^{-(1/\alpha+1)} \\ &= \exp\left\{-\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha}\right\}\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha-1}x^{-(1/\alpha-1)}.\end{aligned}$$

Example 6.1: Solution to (b)(i)

Differentiating the result w.r.t. y then gives (after some algebra)

$$\begin{aligned}\frac{\partial^2 G}{\partial x \partial y} &= (xy)^{-(1/\alpha+1)}\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha-2} \\ &\quad \times\left[\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha}-(1-1/\alpha)\right] \\ &\quad \times\exp\left\{-\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha}\right\} \\ &= (\tilde{x}\tilde{y})^{-(1/\alpha+1)}\left(\tilde{x}^{-1/\alpha}+\tilde{y}^{-1/\alpha}\right)^{\alpha-2} \\ &\quad \times\left[\left(\tilde{x}^{-1/\alpha}+\tilde{y}^{-1/\alpha}\right)^{\alpha}-(1-1/\alpha)\right] \\ &\quad \times\exp\left\{-\left(\tilde{x}^{-1/\alpha}+\tilde{y}^{-1/\alpha}\right)^{\alpha}\right\},\end{aligned}$$

evaluated at $x=\tilde{x}$ and $y=\tilde{y}$.

Example 6.1: Solution to (b)(i)

Region 2:

If only $x > u_x$, then we are in **Region 2** of Figure 6.3, and so

$$g(x, y; \alpha) = \frac{\partial G}{\partial x}.$$

Example 6.1: Solution to (b)(i)

Thus, we have

$$\begin{aligned}\frac{\partial G}{\partial x} &= \exp\left\{-\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha}\right\}\left(x^{-1/\alpha}+y^{-1/\alpha}\right)^{\alpha-1}x^{-(1/\alpha+1)} \\ &= \exp\left\{-\left(\tilde{x}^{-1/\alpha}+\tilde{u}_y^{-1/\alpha}\right)^{\alpha}\right\}\left(\tilde{x}^{-1/\alpha}+\tilde{u}_y^{-1/\alpha}\right)^{\alpha-1}\tilde{x}^{-(1/\alpha+1)}\end{aligned}$$

evaluated at $x = \tilde{x}$ and $y = \tilde{u}_y$.

Region 3:

If only $y > u_y$, then we are in **Region 3** of Figure 6.3, and so

$$g(x, y; \alpha) = \frac{\partial G}{\partial y}.$$

Thus, in a similar fashion, we have:

$$\frac{\partial G}{\partial y} = \exp\left\{-\left(\tilde{u}_x^{-1/\alpha}+\tilde{y}^{-1/\alpha}\right)^{\alpha}\right\}\left(\tilde{u}_x^{-1/\alpha}+\tilde{y}^{-1/\alpha}\right)^{\alpha-1}\tilde{y}^{-(1/\alpha+1)}$$

evaluated at $x = \tilde{u}_x$ and $y = \tilde{y}$.

Region 4:

If both x and y are sub–threshold, then we are in **Region 4** of Figure 6.3. Thus,

$$g(x, y; \alpha) = G(\tilde{u}_x, \tilde{u}_y) = \exp\left\{-\left(\tilde{u}_x^{-1/\alpha} + \tilde{u}_y^{-1/\alpha}\right)^{\alpha}\right\}.$$

(ii) Use the evd function `fbvpot` to fit the logistic model with likelihood contributions you identified in part (i). Show your estimated dependence parameter α, with it's standard error.

[Demo in R*]*

(iii) Using your fitted model in (ii), find the probability that the flood defence system in Newlyn will be overwhelmed. Compare your answer to that in part (a).

Example 6.1: Solution to part (b)(iii)

Assuming extremal dependence, we have

Pr(Flood defence fails) =

$$
\begin{aligned}
& 1 - G\left(\widetilde{9}, \widetilde{0.7}\right) \\
= \quad & 1 - \exp\left\{-\left(\widetilde{9}^{-1/0.764} + \widetilde{0.7}^{-1/0.764}\right)^{0.764}\right\}.
\end{aligned}
$$

Using our estimated marginal GPD parameters from the output above, and the transformation given by Equation (6.4), we have

$$
\widetilde{9} = 321.4721 \qquad \widetilde{0.7} = 1161.417.
$$

Example 6.1: Solution to part (b)(iii)

This gives, on substitution into the above expression,

Pr(Flood defence fails)

$$= \quad 1 - \exp\left\{-\left(321.4721^{-1/0.764} + 1161.417^{-1/0.764}\right)^{0.764}\right\} = 0.00354.$$

Although this probability is small:

- It's more than 2000 times larger than the probability obtained assuming independence!
- Assuming independence gives us a **false sense of security** $\longrightarrow$ **grossly underestimates** the chance of the flood defence being overwhelmed
- This is typical of what happens when we ignore extremal dependence!

Example 6.1: Wave–surge analysis at Newlyn

(c) Check for the presence of asymmetry in the dependence structure.

[Demo in R*]*