SECTION B

B3. A random sample $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$ is taken from a normal $N(\mu, 1/\tau)$ distribution, with sample mean $\bar{x}$ and variance $s^2 = \sum_{i=1}^n (x_i - \bar{x})^2/n$.

The likelihood function is

$$f(\boldsymbol{x}|\mu,\tau) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left[-\frac{n\tau}{2}\left\{s^2 + (\bar{x}-\mu)^2\right\}\right].$$

Suppose your prior beliefs about $(\mu, \tau)^T$ follow a normal-gamma $NGa(b, c, g, h)$ distribution.

(a) Show that your posterior distribution for $(\mu, \tau)^T$ is a $NGa(B, C, G, H)$ distribution where

$$B = \frac{bc + n\bar{x}}{c+n}, \quad C = c+n, \quad G = g+\frac{n}{2}, \quad H = h+\frac{cn(\bar{x}-b)^2}{2(c+n)}+\frac{ns^2}{2}.$$

Hint:

$$c(\mu-b)^2 + n(\bar{x}-\mu)^2 + 2h + ns^2 = C(\mu - B)^2 + 2H.$$

(b) Use this joint posterior distribution to determine your marginal posterior density for μ. Name the posterior distribution for μ.

(c) Suppose Y is the next value randomly sampled from the population. By writing $Y = \mu + \epsilon$, determine the predictive density for Y.

(d) The aerobic capacity of 50 randomly chosen athletes was measured, giving a mean of $\bar{x} = 73.5$ ml/kg/min and standard deviation $s = 5.2$ ml/kg/min. These measurements are assumed to be a random sample from a Normal distribution with mean μ and precision τ.

Work with a sports scientist suggests the following prior distribution:

$$\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(80, 3, 2, 0.5).$$

(i) Obtain the joint posterior distribution for $(\mu, \tau)^T|\boldsymbol{x}$.

(ii) Find a 95% *highest density interval* for the mean aerobic capacity, μ.

(iii) Find a 95% prediction interval for another athlete to be accepted into the sample. Hint: For parts (ii) and (iii) you might find the following R output useful:

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> qt(0.975, 54)
[1] 2.004879
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[30 marks]