Chapter 2

Inference for a Normal population

2.1 Bayes Theorem for many parameters

Data: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$

Model: depends on many parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_p)^T$
pdf/pf $f(\boldsymbol{x}|\boldsymbol{\theta}) \rightarrow$ Likelihood function $f(\boldsymbol{x}|\boldsymbol{\theta})$

Prior beliefs: pdf/pf $\pi(\boldsymbol{\theta})$

Combine using Bayes Theorem

Posterior beliefs: pdf/pf $\pi(\boldsymbol{\theta}|\boldsymbol{x})$

Posterior distribution summarises all our current knowledge about the parameter $\boldsymbol{\theta}$

Bayes Theorem

The posterior probability (density) function for $\boldsymbol{\theta}$ is

$$\pi(\boldsymbol{\theta}|\boldsymbol{x}) = \frac{\pi(\boldsymbol{\theta})\, f(\boldsymbol{x}|\boldsymbol{\theta})}{f(\boldsymbol{x})}$$

where

$$f(\boldsymbol{x}) = \begin{cases} \int_\Theta \pi(\boldsymbol{\theta})\, f(\boldsymbol{x}|\boldsymbol{\theta})\, d\boldsymbol{\theta} & \text{if } \boldsymbol{\theta} \text{ is continuous,} \\ \\ \sum_\Theta \pi(\boldsymbol{\theta})\, f(\boldsymbol{x}|\boldsymbol{\theta}) & \text{if } \boldsymbol{\theta} \text{ is discrete.} \end{cases}$$

As before, this can be rewritten as

$$\pi(\boldsymbol{\theta}|\boldsymbol{x}) \propto \pi(\boldsymbol{\theta}) \times f(\boldsymbol{x}|\boldsymbol{\theta})$$
$$i.e. \quad \text{posterior} \propto \text{prior} \times \text{likelihood}$$

Generalised t distribution: $X \sim t_a(b, c)$

- Density for $x \in \mathbb{R}$

$$f(x|a,b,c) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{ac\pi}\,\Gamma\left(\frac{a}{2}\right)} \left\{1 + \frac{(x-b)^2}{ac}\right\}^{-\frac{a+1}{2}}$$

- Parameters: $a > 0,\ b \in \mathbb{R},\ c > 0$
- $E(X) = Mode(X) = b$ and $Var(X) = \dfrac{ac}{a-2}$, if $a \geq 2$
- Generalisation of the standard t-distribution since $(X-b)/\sqrt{c} \sim t_a$
- $t_a(0,1) \equiv t_a$
- $\lim_{a\to\infty} t_a(b,c) = N(b,c)$

Example 2.1

- If $X \sim t_a(b, c)$ then show that $Y = (X - b)/\sqrt{c} \sim t_a$, with density

$$f(y) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{a\pi}\,\Gamma\left(\frac{a}{2}\right)} \left\{1 + \frac{y^2}{a}\right\}^{-\frac{a+1}{2}}, \qquad -\infty < y < \infty,$$

Recall the general result: if X is a random variable with probability density function $f_X(x)$ and g is a bijective (1–1) function then the random variable $Y = g(X)$ has probability density function

$$f_Y(y) = f_X\left\{g^{-1}(y)\right\} \left|\frac{d}{dy} g^{-1}(y)\right|. \tag{2.1}$$

Solution

. . .

Comments

- R functions `pgt` and `dgt` in the package `nclbayes` give values for $F_X(x)$ and $f_X(x)$ when $X \sim t_a(b, c)$
- Relationship between the generalised t distribution and the standard t distribution is similar to that of the normal distribution and the standard normal distribution:

$$X \sim N(b, c) \quad \Longrightarrow \quad \frac{X - b}{\sqrt{c}} \sim N(0, 1)$$

$$X \sim t_a(b, c) \quad \Longrightarrow \quad \frac{X - b}{\sqrt{c}} \sim t_a$$

Inverse Chi distribution: $Y \sim \mathit{Inv\text{-}Chi}(a, b)$

- Density
$$f(y|a,b) = \frac{2b^a y^{-2a-1} e^{-b/y^2}}{\Gamma(a)}, \quad y > 0$$
- Parameters: $a > 0$, $b > 0$
- $E(Y) = \dfrac{\sqrt{b}\,\Gamma(a-1/2)}{\Gamma(a)}$
- $Var(Y) = \dfrac{b}{a-1} - E(Y)^2$, if $a \geq 1$
- The name of the distribution comes from the fact that $1/Y^2 \sim Ga(a,b) \equiv \chi^2_{2a}/(2b)$

2.2 Prior to posterior analysis

- Data: $X_i|\mu,\tau \sim N(\mu, 1/\tau)$, $i = 1, 2, \ldots, n$ (indep)
- Prior: $\mu|\tau \sim N\left(b, \frac{1}{c\tau}\right)$, $\tau \sim Ga(g, h)$, with joint density, for $\mu \in \mathbb{R}$, $\tau > 0$

$$
\begin{aligned}
\pi(\mu,\tau) &= \pi(\mu|\tau)\pi(\tau) \\
&= \left(\frac{c\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{c\tau}{2}(\mu-b)^2\right\} \times \frac{h^g \tau^{g-1} e^{-h\tau}}{\Gamma(g)} \\
&\propto \tau^{g-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[c(\mu-b)^2 + 2h\right]\right\} \qquad (2.2)
\end{aligned}
$$

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\end{aligned}$$

- We write $\begin{pmatrix}\mu \\ \tau\end{pmatrix} \sim NGa(b, c, g, h)$
- Determine the posterior distribution for $\begin{pmatrix}\mu \\ \tau\end{pmatrix}$

  Hint:

$$c(\mu - b)^2 + n(\bar{x} - \mu)^2 = (c + n)\left\{\mu - \left(\frac{cb + n\bar{x}}{c + n}\right)\right\}^2 + \frac{nc(\bar{x} - b)^2}{c + n}$$

Solution

$$\cdots \tag{2.3}$$

2.2.1 Marginal distributions

- If $\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(b, c, g, h)$ then $\tau \sim Ga(g, h)$
- The (marginal) density for μ is, for $\mu \in \mathbb{R}$

$$\begin{aligned}
\pi(\mu) &= \int_0^\infty \pi(\mu, \tau)\, d\tau \\
&\propto \int_0^\infty \tau^{g-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[c(\mu-b)^2+2h\right]\right\} d\tau \\
&\propto \int_0^\infty \tau^{g+\frac{1}{2}-1} \exp\left\{-\frac{\tau}{2}\left[c(\mu-b)^2+2h\right]\right\} d\tau \\
&\propto \frac{\Gamma\left(g+\frac{1}{2}\right)}{[\{c(\mu-b)^2+2h\}/2]^{g+\frac{1}{2}}} \quad \text{using } \int_0^\infty \theta^{a-1}e^{-b\theta}\, d\theta = \frac{\Gamma(a)}{b^a} \\
&\propto h^{-g-1/2}\left\{1+\frac{c(\mu-b)^2}{2h}\right\}^{-g-1/2} \\
&\propto \left\{1+\frac{c(\mu-b)^2}{2h}\right\}^{-\frac{2g+1}{2}} \\
&\text{i.e.} \quad \mu \sim t_{2g}\left(b, \frac{h}{gc}\right) \qquad (2.4)
\end{aligned}$$

Summary of marginal distributions

- The prior $\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(b, c, g, h)$ has marginal distributions
  - $\mu \sim t_{2g}\left(b, \frac{h}{gc}\right)$
  - $\tau \sim Ga(g, h)$

  Also $\sigma = 1/\sqrt{\tau} \sim \textit{Inv-Chi}(g, h)$

- The posterior $\left.\begin{pmatrix} \mu \\ \tau \end{pmatrix}\right| \boldsymbol{x} \sim NGa(B, C, G, H)$ has marginal distributions
  - $\mu|\boldsymbol{x} \sim t_{2G}\left(B, \frac{H}{GC}\right)$
  - $\tau|\boldsymbol{x} \sim Ga(G, H)$

  Also $\sigma|\boldsymbol{x} \sim \textit{Inv-Chi}(G, H)$

Example 2.2

- The 18th century physicist Henry Cavendish made 23 experimental determinations of the earth's density, and these data (in g/cm^3) are

5.36	5.29	5.58	5.65	5.57	5.53	5.62	5.29
5.44	5.34	5.79	5.10	5.27	5.39	5.42	5.47
5.63	5.34	5.46	5.30	5.78	5.68	5.85	

  with sufficient statistics $n = 23$, $\bar{x} = 5.4848$, $s = 0.1882$
- We consider $X|\mu, \tau \sim N(\mu, 1/\tau)$, with τ unknown
- In Example 1.4 we assumed $X_i|\mu \sim N(\mu, 0.2^2)$ with $\mu \sim N(5.41, 0.4^2)$ and τ known with $\tau = 1/0.2^2 = 25$.
- Must specify the parameters in the $NGa(b, c, g, h)$ prior distribution for $(\mu, \tau)^T$. Suppose we choose $Var(\tau) = 250$.
- Choice of b and c: the NGa prior distribution has $\mu|\tau \sim N\{b, 1/(c\tau)\}$. Matching the prior for $\mu|\tau = 25$ gives $b = 5.41$ and $c = 0.25$

  Choice of g and h: the NGa prior distribution has $\tau \sim Ga(g, h)$. Choose $E(\tau) = 25$, giving $g = 2.5$ and $h = 0.1$

- Summary: take prior

$$\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(b = 5.41, c = 0.25, g = 2.5, h = 0.1)$$

- Is the marginal prior distribution for μ close to what we used in Example 1.4? Yes, see plot below

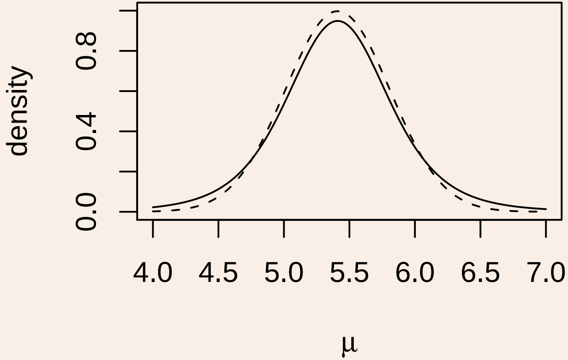


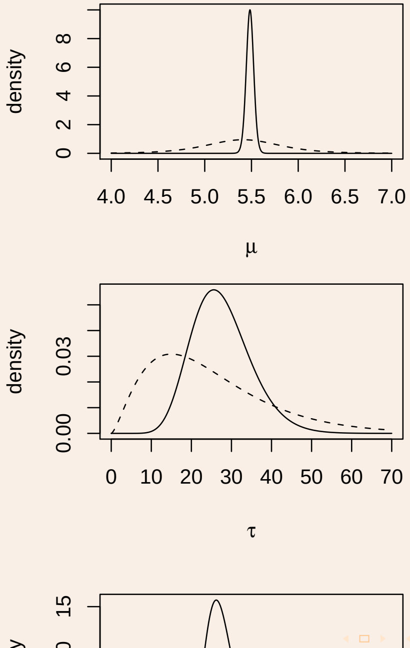
Figure: Marginal prior density for μ: new version (solid) and previous version (dashed)

- Determine the posterior distribution for $(\mu, \tau)^T$. Also determine the marginal prior distribution for τ and for σ, and the marginal posterior distribution for each of μ, τ and σ.

Solution

$\cdots$

Comparison of prior and posterior marginal distributions



Review of contour plots for bivariate distribution

- Bivariate normal: $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \right\}$, with density $\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} \right) \right\}$.

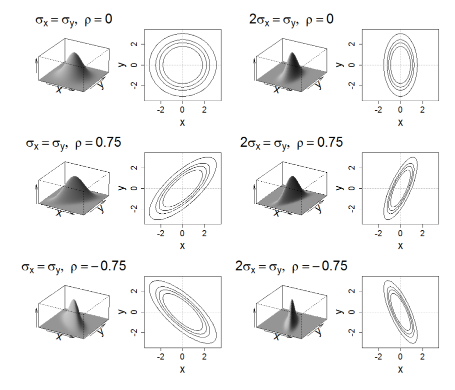


Figure: Contour plots for different bivariate normal densities

Comparison of prior and posterior distributions

- Can plot their contours using R command NGacontour:

```
mu=seq(4.5,6.5,len=1000); tau=seq(0,71,len=1000)
NGacontour(mu,tau,b,c,g,h,lty=3); NGacontour(mu,tau,B,C,G,H,add=TRUE)
```

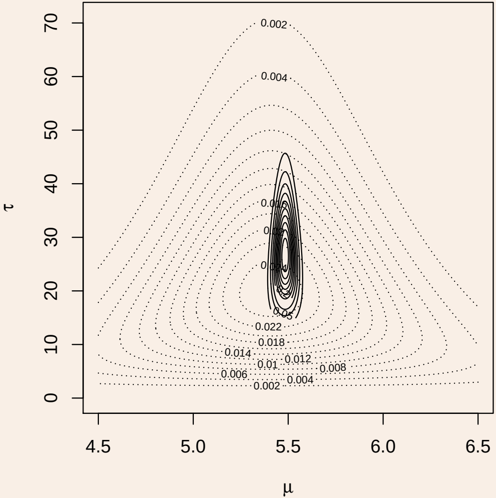


Figure: Contour plot of the prior (dashed) and posterior (solid) densities for $(\mu, \tau)^T$.

Comments

- Wikipedia tells us that the actual mean density of the earth is $5.515\,g/cm^3$
- What is the (posterior) probability that the mean density is within 0.1 of this value?

$$\mu|\boldsymbol{x} \sim t_{28}(5.484, 0.001561) \quad \Rightarrow \quad Pr(5.415 < \mu < 5.615|\boldsymbol{x}) = 0.9529$$

using

```
pgt(5.615,28,5.484,0.001561)-pgt(5.415,28,5.484,0.001561)
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  using
  `pgt(5.615,28,5.484,0.001561)-pgt(5.415,28,5.484,0.001561)`
- Without the data, the only basis for determining the earth's density is via the prior distribution

$$\mu \sim t_5(5.41, 0.16) \quad \Rightarrow \quad Pr(5.415 < \mu < 5.615) = 0.1802$$

  using `pgt(5.615,5,5.41,0.16)-pgt(5.415,5,5.41,0.16)`
- These probability calculations demonstrate that the data have been very informative and changed our beliefs about the earth's density

Example 2.3

Determine the $100(1-\alpha)\%$ highest density interval (HDI) for the population mean μ in terms of quantiles of the standard t-distribution

Solution

. . .

Calculating 95% posterior confidence intervals using R

- $\mu|\boldsymbol{x} \sim t_{2G}\{B, H/(GC)\}$
  Symmetric distribution $\rightarrow$ HDI and equi-tailed intervals are the same
  `c(qgt(0.025,2*G,B,H/(G*C)),qgt(0.975,2*G,B,H/(G*C)))`

- $\tau|\boldsymbol{x} \sim Ga(G, H)$
  Skewed distribution $\rightarrow$ HDI and equi-tailed intervals are different
  `hdiGamma(0.95,G,H)`
  `c(qgamma(0.025,G,H),qgamma(0.975,G,H))`

- $\sigma|\boldsymbol{x} \sim \textit{Inv-Chi}(G, H)$
  Skewed distribution $\rightarrow$ HDI and equi-tailed intervals are different
  `hdiInvchi(0.95,G,H)`
  `c(qinvchi(0.025,G,H),qinvchi(0.975,G,H))`

Results from this data analysis ...

	Prior	Posterior	
μ:	$(4.3818, 6.4382)$	$(5.4031, 5.5649)$	
τ:	$(1.4812, 55.9573)$	$(14.0193, 42.2530)$	$\leftarrow$ HDI
	$(4.1561, 64.1625)$	$(15.0674, 43.7625)$	
σ:	$(0.1062, 0.4246)$	$(0.1466, 0.2505)$	$\leftarrow$ HDI
	$(0.1248, 0.4905)$	$(0.1512, 0.2576)$	

- Posterior HDI and equi-tailed intervals for τ are fairly similar but prior intervals are not
- Ditto for σ
- Why?

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- Ditto for σ
- Why?

  The prior distributions are quite skewed but the posterior distributions are fairly symmetric

Confidence regions

- We have looked at (marginal) HDIs
- Can be useful to also look at (joint) confidence regions

Example 2.4

Determine a joint α confidence region for $(\mu, \tau)^T$ by calculating k satisfying $P(\pi(\mu,\tau) > k) = \alpha$ or $P(\pi(\mu,\tau) > k \mid \boldsymbol{x}) = \alpha$.

Solution

. . .

- Can plot these regions using `NGacontour` by using the appropriate value for the contour level

```
mu=seq(3.5,7.5,len=1000)
tau=seq(0,80,len=1000)
NGacontour(mu,tau,b,c,g,h,p=c(0.95,0.9,0.8),lty=3)
NGacontour(mu,tau,B,C,G,H,p=c(0.95,0.9,0.8),add=TRUE)
```

Confidence regions for $(\mu, \tau)^T$

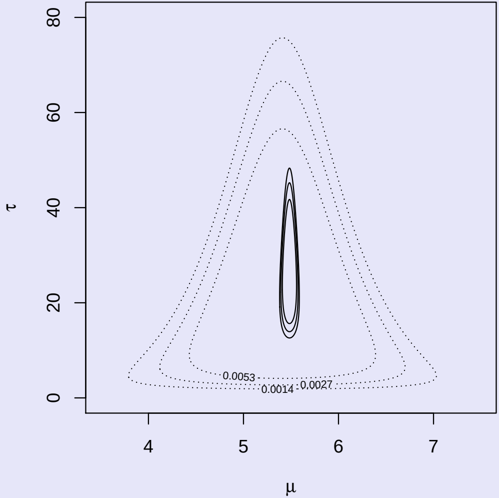
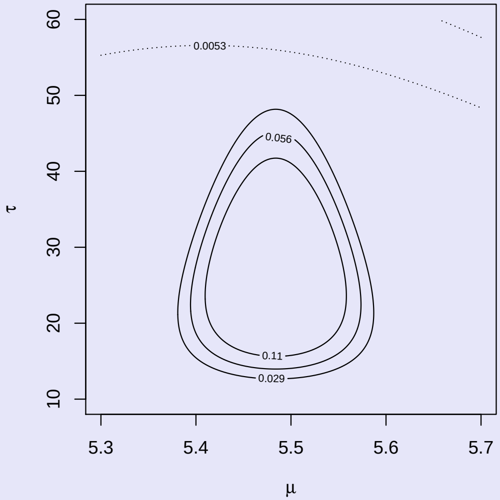


Figure: 95%, 90% and 80% prior (dashed) and posterior (solid) confidence regions for $(\mu, \tau)^T$; 95% (outer), 80% (inner).

Focusing on central part of plot . . .



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Mid-term feedback: A snapshot of your comments

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- What could be improved? The shirts

2.4 Predictive distributions

- In this model we can determine the predictive distribution via

$$f(y|\boldsymbol{x}) = \int f(y|\mu,\tau)\,\pi(\mu,\tau|\boldsymbol{x})\,d\mu\,d\tau$$

  or by using Candidate's formula (as this is a conjugate analysis)
- But, for this model, there is a more straightforward way

. . .

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- But, for this model, there is a more straightforward way

. . .

- These predictive intervals can be calculated easily using the R function qgt
- In Example 3.2, the prior and posterior predictive HDIs for a new value Y from the population are $(4.2604, 6.5596)$ and $(5.0855, 5.8825)$ respectively, calculated using

```
c(qgt(0.025,2*g,b,h*(c+1)/(g*c)),qgt(0.975,2*g,b,h*(c+1)/(g*c)))
c(qgt(0.025,2*G,B,H*(C+1)/(G*C)),qgt(0.975,2*G,B,H*(C+1)/(G*C)))
```

2.5 Summary

Suppose we have a normal random sample with $X_i|\mu,\tau \sim N(\mu, 1/\tau)$, $i = 1, 2, \ldots, n$ (independent)

1. $(\mu,\tau)^T \sim NGa(b, c, g, h)$ is a conjugate prior distribution
2. The posterior distribution is $(\mu,\tau)^T|\boldsymbol{x} \sim NGa(B, C, G, H)$ where the posterior parameters are given by (2.3)
3. The marginal prior distributions are $\mu \sim t_{2g}\{b, h/(gc)\}$, $\tau \sim Ga(g, h)$, $\sigma = 1/\sqrt{\tau} \sim \textit{Inv-Chi}(g, h)$
4. The marginal posterior distributions are $\mu|\boldsymbol{x} \sim t_{2G}\{B, H/(GC)\}$, $\tau|\boldsymbol{x} \sim Ga(G, H)$, $\sigma|\boldsymbol{x} \sim \textit{Inv-Chi}(G, H)$
5. Prior and posterior means and standard deviations for μ, τ and σ can be calculated from the properties of the t, *Gamma* and *Inv-Chi* distributions
6. Prior and posterior probabilities and densities for μ, τ and σ can be calculated using the `R` functions `pgt`, `dgt`, `pgamma`, `dgamma`, `pinvchi`, `dinvchi`

Suppose we have a normal random sample with $X_i|\mu,\tau \sim N(\mu, 1/\tau)$, $i = 1, 2, \dots, n$ (independent)

(vii) HDIs or equi-tailed CIs for μ, τ and σ can be calculated using `qgt`, `hdiGamma`, `hdiInvchi`, `qgamma`, `qinvchi`

(viii) Contour plots of the prior and posterior densities for $(\mu,\tau)^T$ can be plotted using the `NGacontour` function

(ix) Prior and posterior confidence regions for $(\mu,\tau)^T$ can be plotted using the `NGacontour` function

(x) The predictive distribution for a new observation Y from the population is $Y|\boldsymbol{x} \sim t_{2G}\{B, H(C+1)/(GC)\}$ and its HDI can be calculated using the `qgt` function

2.6 Why do we have so many different distributions?

- So far we have used many distributions, some you will have met before and some will be new
- After a while the variety and sheer number of different distributions can become overwhelming
- Why do we need so many distributions and why do we name so many of them?

Justification

- Statistics studies the random variation in experiments, samples and processes
- The variety of applications leads to their randomness being described by many different distributions
- In many applications, bespoke distributions need to be formulated
- However, some distributions come up time and time again for modelling random variation in data and for describing prior beliefs

Justification (continued)

- It is helpful for us to be able to refer to these distributions – and so we give each one a name – and be able to quote known results for these distributions such as their mean and variance
- For example, in this chapter you have been introduced to
  - a generalisation of the t-distribution
  - the inverse chi distribution
  - The Normal-Gamma distribution
- We have been able to use results for their mean and variance to study prior and posterior distributions and have been able to plot these distributions using functions in the R package
- You will meet several other new distributions in the remainder of the module

Justification (continued)

- Obviously it's useful to have a working knowledge of each of these distributions but not vital to remember all their properties

Justification (continued)

- Obviously it's useful to have a working knowledge of each of these distributions but not vital to remember all their properties
- The exam paper will contain a list of all the distributions used in the exam, together with their density (or probability function) and any useful proprieties such as their mean and variance (as needed for the exam)

2.7 Learning objectives

By the end of this chapter, you should be able to:

1. Determine the posterior distribution for $(\mu, \tau)^T$
2. Determine and use the univariate prior and posterior distributions
3. Determine confidence intervals, HDIs and confidence regions
4. Determine the predictive distribution of another value from the population, and its predictive interval
5. Determine the predictive distribution of the mean of another random sample from the population

both in general and for a particular prior and data set.

Also you should be able to:

6. Appreciate the benefit of naming distributions and for having lists of properties for these distributions