MAS3902 - SPECIMEN PAPER 1

A1. We have n inter-event times between the first $n+1$ accidents, giving $X_i|\theta \sim Exp(\theta)$, $i=1,2,\ldots,n$.

(a) $f(\underline{x}|\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} \propto \theta^n e^{-\theta \Sigma x_i} = \theta^n e^{-n\bar{x}\theta}$, $\theta > 0$.

(b) $\theta \sim Ga(g,h) \Rightarrow \pi(\theta) \propto \theta^{g-1} e^{-h\theta}$, $\theta > 0$.

so

$$\pi(\theta|\underline{x}) \propto \theta^{g-1} e^{-h\theta} \times \theta^n e^{-n\bar{x}\theta} \propto \theta^{(g+n)-1} e^{-(h+n\bar{x})\theta}, \quad \theta > 0$$

so $\theta|\underline{x} \sim Ga(g+n, h+n\bar{x})$.

(c) Yes — both prior and posterior are from the same family.

(d) Numerator: $y|\theta \sim Exp(\theta)$ and $\theta|\underline{x} \sim Ga(g+n, h+n\bar{x})$

Denominator: Treat our "old posterior" as the "new prior". We know that the "new posterior" will be $Ga(g^*+n^*, h^*+n^*\bar{y})$, with $g^* = g+n$ and $h^* = h+n\bar{x}$. But what about n^* and $\bar{y}$? We are interested in the predictive distribution of a single new inter-event time, and so $n^*=1$ and $\bar{y}=y$, giving $\theta|\underline{x},y \sim Ga(g+n+1, h+n\bar{x}+y)$.

Therefore,

$$f(y|\underline{x}) = \frac{\theta e^{-\theta y} \cdot \dfrac{(h+n\bar{x})^{g+n}\, \theta^{g+n-1}\, e^{-(h+n\bar{x})\theta}}{\Gamma(g+n)}}{\dfrac{(h+n\bar{x}+y)^{g+n+1}\, \theta^{g+n}\, e^{-(h+n\bar{x}+y)\theta}}{\Gamma(g+n+1)}}$$

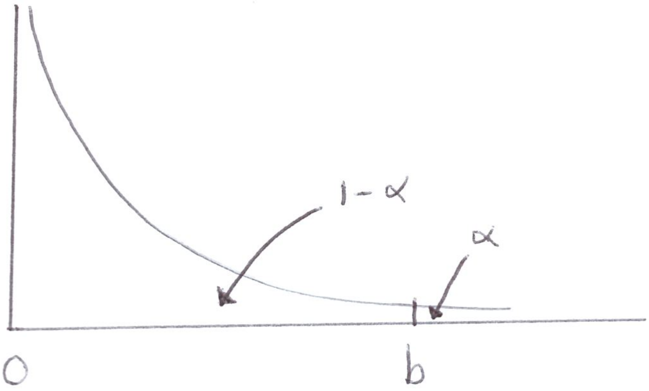
$$= \frac{(h+n\bar{x})^{g+n}}{(h+n\bar{x}+y)^{g+n+1}} \times \frac{\Gamma(g+n+1)}{\Gamma(g+n)} \times \frac{\cancel{\theta^{g+n}}}{\cancel{\theta^{g+n}}} \times \frac{\cancel{e^{-(h+n\bar{x}+y)\theta}}}{\cancel{e^{-(h+n\bar{x}+y)\theta}}}$$

$$= \frac{(h+n\bar{x})^{g+n}}{(h+n\bar{x}+y)^{g+n+1}} \times \frac{(g+n)!}{(g+n-1)!}$$

so $$f(y|\underline{x}) = \frac{(g+n)(h+n\bar{x})^{g+n}}{(h+n\bar{x}+y)^{g+n+1}} = \frac{G \cdot H^{G}}{(H+y)^{G+1}}, \quad y >$$

with $G = g+n$ and $H = h+n\bar{x}$.

(e) The predictive density will take the form:



And so the prediction interval is $(0, b)$, where

$$\int_0^b \frac{G H^G}{(H+y)^{G+1}}\, dy = 1-\alpha$$

Using $Z = H+y$ gives

$$\int_0^b \frac{G H^G}{(H+y)^{G+1}}\, dy = \int_H^{b+H} \frac{G \cdot H^G}{Z^{G+1}}\, dZ$$

$$= \left[-\frac{H^G}{Z^G} \right]_H^{b+H} = 1 - \frac{H^G}{(b+H)^G}$$

So we require $1 - \frac{H^G}{(b+H)^G} = 1-\alpha \Rightarrow b = H(\alpha^{-1/G} - 1$

giving our $100(1-\alpha)\%$ prediction interval as

$$(0, H(\alpha^{-1/G} - 1)).$$

$\boxed{A2}$. From Page 7, we have

$$f(x|\mu,\sigma^2) = \frac{1}{x\sigma\sqrt{2\pi}} \cdot exp\left\{-\frac{1}{2\sigma^2}(\log x-\mu)^2\right\}.$$

(a) $$f(\underline{x}|\mu,\sigma^2) = \prod_{i=1}^{n} \frac{1}{x_i\sigma\sqrt{2\pi}} \cdot exp\left\{-\frac{1}{2\sigma^2}(\log x_i-\mu)^2\right\}$$

$$= \left(\prod_{i=1}^{n}\frac{1}{x_i}\right) \times \left(\frac{1}{\sqrt{2\pi}}\right)^n \times \left(\frac{1}{\sigma}\right)^n \times exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i-\mu)^2\right\}$$

$$\propto \sigma^{-n} \cdot exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i-\mu)^2\right\}$$

(b) $$l = \log f(\underline{x}|\mu,\sigma^2) = -n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i-\mu)^2$$

$$\frac{\partial l}{\partial\mu} = -\frac{1}{2\sigma^2}\times 2\sum_{i=1}^{n}(\log x_i-\mu)^1 \times(-1) \quad \text{(Chain rule)}$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(\log x_i-\mu) \quad \left[= \frac{1}{\sigma^2}\left(\sum\log x_i - n\mu\right)\right]$$

$$\Rightarrow \frac{\partial^2 l}{\partial\mu^2} = -\frac{n}{\sigma^2}.$$

$$\frac{\partial l}{\partial\sigma} = -\frac{n}{\sigma} + \frac{2}{2}\sigma^{-3}\sum_{i=1}^{n}(\log x_i-\mu)^2$$

$$= -\frac{n}{\sigma} + \sigma^{-3}\sum_{i=1}^{n}(\log x_i-\mu)^2$$

$$\Rightarrow \frac{\partial^2 l}{\partial\sigma^2} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\cdot\sum_{i=1}^{n}(\log x_i-\mu)^2$$

$$\Rightarrow \frac{\partial^2 l}{\partial\mu\partial\sigma} = \frac{\partial}{\partial\sigma}\left(\frac{\partial l}{\partial\mu}\right) = \frac{\partial}{\partial\sigma}\left(\sigma^{-2}\cdot\sum_{i=1}^{n}(\log x_i-\mu)\right)$$

$$= -\frac{2}{\sigma^3}\sum_{i=1}^{n}(\log x_i-\mu)$$

(c) The Jeffreys Prior is given by

$$\pi(\mu,\sigma^2) = \sqrt{\det(I(\mu,\sigma))}.$$

$$I_{11}(\mu,\sigma) = E_{X|\mu,\sigma}\left[-\frac{\partial^2 l}{\partial\mu^2}\right] = E_{X|\mu,\sigma}\left[\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}.$$

$$I_{22}(\mu,\sigma) = E_{X|\mu,\sigma}\left[-\frac{\partial^2 l}{\partial\sigma^2}\right]$$

$$= E_{X|\mu,\sigma}\left[\frac{-n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n}(\log X_i-\mu)^2\right]$$

$$= -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\cdot E_{X|\mu,\sigma}\left[\sum_{i=1}^{n}(\log X_i-\mu)^2\right]$$

$$= -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\cdot E_{X|\mu,\sigma}\left[n.\,Var(\log X_i)\right]$$

$$= -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\cdot n\,E_{X|\mu,\sigma}\left[Var(\log X_i)\right]$$

$$= -\frac{n}{\sigma^2} + \frac{3n}{\sigma^4}\cdot E_{X|\mu,\sigma}\left[\sigma^2\right] \quad \text{(using the Hint)}$$

$$= -\frac{n}{\sigma^2} + \frac{3n\cancel{\sigma^2}}{\sigma^{\cancel{4}2}} = -\frac{n}{\sigma^2} + \frac{3n}{\sigma^2} = \frac{2n}{\sigma^2}.$$

$$I_{12}(\mu,\sigma) = I_{21}(\mu,\sigma) = E_{X|\mu,\sigma}\left[\frac{2}{\sigma^3}\cdot\sum_{i=1}^{n}(\log X_i-\mu)\right]$$

$$= \frac{2}{\sigma^3}E_{X|\mu,\sigma}\left[\sum_{i=1}^{n}\log X_i - \sum_{i=1}^{n}\mu\right]$$

$$= \frac{2}{\sigma^3}E_{X|\mu,\sigma}\left[\sum_{i=1}^{n}\log X_i - n\mu\right]$$

$$= \frac{2}{\sigma^3}\left\{E_{X|\mu,\sigma}\left[\sum_{i=1}^{n}\log X_i\right] - E_{X|\mu,\sigma}\left[n\mu\right]\right\}$$

$$= \frac{2}{\sigma^3}\left\{E_{X|\mu,\sigma}[\log X_1]+\dots+E_{X|\mu,\sigma}[\log X_n] - n\mu\right\}$$

$$= \frac{2}{\sigma^3}\left\{\mu+\mu+\dots+\mu - n\mu\right\} \quad \text{(using the Hint)}$$

$$= \frac{2}{\sigma^3}\left\{n\mu - n\mu\right\} = 0.$$

$$\Rightarrow I = \begin{pmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{pmatrix}$$

$$\therefore \pi(\mu,\sigma^2) = \sqrt{n/\sigma^2 \times 2n/\sigma^2} = \sqrt{\frac{2n^2}{\sigma^4}} = \frac{(\sqrt{2})n}{\sigma^2} \propto \frac{1}{\sigma^2}.$$

(d) NOT a proper prior distribution, as

$$\int_0^\infty \int_{-\infty}^\infty \frac{1}{\sigma^2} \, d\mu \, d\sigma$$

is a divergent integral.

(e) YES – the (joint) prior depends only on σ, and we can write

$$\pi(\mu, \sigma^2) \propto \frac{1}{\sigma^2} \times 1 = \pi(\sigma) \times \pi(\mu).$$

B3. (a) We have

$$\begin{aligned}
\pi(\mu, \tau | \underline{x}) &\propto \pi(\mu, \tau) \times f(\underline{x} | \mu, \tau) \\
&\propto \tau^{g-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[c(\mu-b)^2 + 2h\right]\right\} \times \tau^{n/2} \exp\left\{-\frac{n\tau}{2}\left(s^2 + (\bar{x}-\mu)^2\right)\right\} \\
&\propto \tau^{g+n/2-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[c(\mu-b)^2 + n(\bar{x}-\mu)^2 + 2h + ns^2\right]\right\} \\
&\propto \tau^{G-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[C(\mu-B)^2 + 2H\right]\right\}, \quad \mu \in \mathbb{R}, \tau > 0.
\end{aligned}$$

Therefore,

$$\binom{\mu}{\tau} \Big| \underline{x} \sim NGa(B, C, G, H).$$

(b) The marginal posterior for μ is, for $\mu \in \mathbb{R}$,

$$\pi(\mu | \underline{x}) = \int_0^\infty \pi(\mu, \tau | \underline{x}) \, d\tau \propto \int_0^\infty \tau^{G-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[C(\mu-B)^2 + 2H\right]\right.$$

Now, as the integral of a gamma density over its entire range is 1, we have

$$\int_0^\infty \theta^{a-1} e^{-b\theta} \, d\theta = \frac{\Gamma(a)}{b^a}.$$

Therefore, for $\mu \in \mathbb{R}$,

$$\pi(\mu | \underline{x}) \propto \int_0^\infty \tau^{G+\frac{1}{2}-1} \exp\left\{-\frac{\tau}{2}\left[C(\mu-B)^2 + 2H\right]\right\} d\tau$$

Giving

$$\pi(\mu|\underline{x}) \propto \frac{\Gamma(G+\frac{1}{2})}{\left[\{C(\mu-B)^2+2H\}/2\right]^{G+\frac{1}{2}}}$$

$$\propto H^{-G-\frac{1}{2}}\left\{1+\frac{C(\mu-B)^2}{2H}\right\}^{-G-\frac{1}{2}}$$

$$\propto \left\{1+\frac{C(\mu-B)^2}{2H}\right\}^{-\frac{2G+1}{2}}$$

Thus, $\mu|\underline{x} \sim t_{2G}\left(B, \frac{H}{GC}\right)$.

(c) $Y=\mu+\varepsilon$, where $\varepsilon|\tau \sim N(0, \frac{1}{\tau})$ and $\mu|\underline{x},\tau \sim N(B, \frac{1}{C\tau})$.

Therefore

$$Y|\underline{x},\tau \sim N\left(B, \frac{1}{\tau}+\frac{1}{C\tau}\right) \equiv N\left(B, \frac{C+1}{C\tau}\right).$$

Thus, as $\tau|\underline{x} \sim Ga(G, H)$,

$$\binom{Y}{\tau}\Big|\underline{x} \sim NGa\left(B, \frac{C}{C+1}, G, H\right)$$

and so, by comparing to the solution to part (b),

$$Y|\underline{x} \sim t_{2G}\left(B, \frac{H(C+1)}{GC}\right).$$

(d) From the prior: $b=80$, $c=3$, $g=2$, $h=0.5$

From the data: $n=50$, $\bar{x}=73.5$, $s=5.2$

(i) Therefore

$$B = \frac{(80\times3)+(50\times73.5)}{53} = 73.868$$

$$C = 3+50 = 53$$

$$G = 2+\frac{50}{2} = 27$$

$$H = 0.5+\frac{150(73.5-80)^2}{2(3+50)}+\frac{50\times5.2^2}{2} = 736.288.$$

$$\Rightarrow \binom{\mu}{\tau}\Big|\underline{x} \sim NGa(73.868, 53, 27, 736.288)$$

(ii) From (b), we know that

$$\mu|\underline{x} \sim t_{2G}\left(B, \frac{H}{GC}\right) \equiv t_{54}(73.868,\ 0.515)$$

"standardising" puts us on the scale of the "standard" (Student's) t-distribution, giving an HDI of

$$B \pm t_{2G} \times \sqrt{\frac{H}{GC}} \quad \text{(owing to the symmetry of the t-distribut}$$

$$= 73.868 \pm t_{54} \times \sqrt{0.515}$$

$$= 73.868 \pm 2.004879 \times \sqrt{0.515} = (72.43, 75.31) \text{ ml/kg/min}$$

(iii) Similarly and from (c),

$$Y|\underline{x} \sim t_{2G}\left(B, \frac{H(C+1)}{GC}\right) \equiv t_{54}(73.868,\ 27.784),$$

giving

$$B \pm t_{2G} \times \sqrt{\frac{H(C+1)}{GC}}$$

$$= 73.868 \pm 2.004879 \times \sqrt{27.784} = (63.300, 84.44) \text{ ml/k}$$

$\boxed{\text{B4}}$ (a) Posterior $\propto$ prior $\times$ likelihood

$$\Rightarrow \pi(\alpha,\lambda|\underline{x}) \propto \pi(\alpha) \times \pi(\lambda) \times f(\underline{x}|\alpha,\lambda)$$

$$\propto \frac{b^a \cdot \alpha^{a-1} \cdot e^{-b\alpha}}{\Gamma(a)} \times \frac{d^c \cdot \lambda^{c-1} \cdot e^{-d\lambda}}{\Gamma(c)} \times \frac{\lambda^{n\alpha} \cdot \exp\{\alpha n\bar{x} - \lambda n\bar{x}_{exp}\}}{\Gamma(\alpha)^n}$$

$$\propto \frac{\alpha^{a-1} \cdot \lambda^{n\alpha+c-1} \cdot \exp\{\alpha(n\bar{x}-b) - \lambda(n\bar{x}_{exp}+d)\}}{\Gamma(\alpha)^n}, \quad \alpha,\lambda > 0$$

(b)
$$\pi(\alpha|\lambda,\underline{x}) \propto \frac{\alpha^{a-1} \cdot \lambda^{n\alpha} \cdot \exp\{\alpha(n\bar{x}-b)\}}{\Gamma(\alpha)^n}$$

$$\propto \frac{\alpha^{a-1} \cdot \exp\{-[(b-n\bar{x}) - n\log\lambda]\alpha\}}{\Gamma(\alpha)^n}, \quad \alpha,\lambda>0.$$

(ALMOST A GAMMA, BUT NOT QUITE, BECAUSE OF THE DENOMINATOR...)

$$\pi(\lambda|\alpha,\underline{x}) \propto \lambda^{n\alpha+c-1} \exp\{-(n\bar{x}_{exp}+d)\lambda\}, \quad \alpha,\lambda>0$$

$$\Rightarrow \lambda|\alpha,\underline{x} \sim Ga(n\alpha+c,\ n\bar{x}_{exp}+d).$$

(c) We have that $\lambda|\alpha,\underline{x} \sim Ga(n\alpha+c, n\bar{x}_{exp}+d)$ but the distribution of $\alpha|\lambda,\underline{x}$ is non-standard. Therefore a Metropolis-within-Gibbs scheme is appropriate, with a Gibbs step for λ and a M-H step for α.

(d) Acceptance probability is $\min(1, A)$, where:

$$A = \frac{\pi(\alpha^*|\lambda,\underline{x})}{\pi(\alpha|\lambda,\underline{x})}$$

$$= \frac{\alpha^{*\,a-1} \exp\{-[(b-n\bar{x})-n\log\lambda]\alpha^*\}}{\Gamma(\alpha^*)^n} \times \frac{\Gamma(\alpha)^n}{\alpha^{a-1} \exp\{-[(b-n\bar{x})-n\log\lambda]\alpha\}}$$

$$= \left(\frac{\alpha^*}{\alpha}\right)^{a-1} \times \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha^*)}\right)^n \times \exp\{-[(b-n\bar{x})-n\log\lambda](\alpha^*-\alpha)\},$$

$\alpha^*>0$ (0 otherwise).

(e) ① Initialise iteration counter to $j=1$, and initialise the chain to $\alpha^{(0)} = \frac{a}{b}$ (prior mean)

② Obtain a new value $\lambda^{(j)} \sim Ga(n\alpha^{(j-1)}+c,\ n\bar{x}_{exp}+d)$ using the available simulator.

③ Generate a proposed value $\alpha^* \sim N(\alpha^{(j-1)}, \Sigma_\alpha)$

④ Evaluate the acceptance probability $\min(1,A)$ at $\alpha^*=\alpha^*, \alpha=\alpha^{(j-1)}$, $\lambda=\lambda^{(j)}$

⑤ Set $\alpha^{(j)}=\alpha^*$ with probability $\min(1,A)$, and set $\alpha^{(j)}=\alpha^{(j-1)}$ otherwise

⑥ Change the counter from j to $j+1$ and return to step ②.