NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS, STATISTICS & PHYSICS

SEMESTER 1 2019/20

MAS3902: Specimen Paper 1

Bayesian Inference

Time allowed: 2 hours

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks may be adjusted in accordance with the University's Moderation and Scaling Policy.

There are TWO questions in Section A and TWO questions in Section B.

Calculators may be used.

SECTION A

A1. Suppose that accidents at a point on a motorway follow a Poisson process with rate θ, so that the times between accidents are independent and follow an exponential $Exp(\theta)$ distribution.

(a) Suppose that the times between $n+1$ accidents gives inter-event times $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$. Show that the likelihood function for θ given these times is

$$f(\boldsymbol{x}|\theta) \propto \theta^n e^{-n\bar{x}\theta}, \quad \theta > 0,$$

where $\bar{x}$ is the mean time between accidents.

[**4 marks**]

(b) Suppose that your prior beliefs about θ were described by a $Ga(g, h)$ distribution. Determine your posterior density for θ. Name this distribution, including its parameters.

[**5 marks**]

(c) Is the *Gamma* distribution a conjugate prior distribution? Explain your answer.

[**2 marks**]

(d) The local authority is interested in the time to the next event, that is, the time Y between accidents $n+1$ and $n+2$. Using Candidate's formula

$$f(y|\boldsymbol{x}) = \frac{f(y|\theta)\pi(\theta|\boldsymbol{x})}{\pi(\theta|\boldsymbol{x}, y)},$$

show that the predictive density for Y is

$$f(y|\boldsymbol{x}) = \frac{GH^G}{(H+y)^{G+1}}, \quad y > 0.$$

[**4 marks**]

(e) Sketch this predictive density and hence determine the $100(1-\alpha)\%$ predictive interval for Y.

[**5 marks**]

[**Total: 20 marks**]

A2. The wages X of a random sample of n investment bankers follow a log-normal $LN(\mu, \sigma^2)$ distribution.

(a) Show that the likelihood function is

$$f(\boldsymbol{x}|\mu, \alpha) \propto \sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\log x_i - \mu)^2\right\}.$$

[2 marks]

(b) Find the log-likelihood function $\log f(\boldsymbol{x}|\mu, \sigma)$, and show that

$$\begin{aligned}
\frac{\partial^2}{\partial\mu^2}\log f(\boldsymbol{x}|\mu,\sigma) &= -\frac{n}{\sigma^2}, \\
\frac{\partial^2}{\partial\sigma^2}\log f(\boldsymbol{x}|\mu,\sigma) &= \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}(\log x_i - \mu)^2, \quad \text{and} \\
\frac{\partial^2}{\partial\mu\partial\sigma}\log f(\boldsymbol{x}|\mu,\sigma) &= -\frac{2}{\sigma^3}\sum_{i=1}^{n}(\log x_i - \mu).
\end{aligned}$$

[6 marks]

(c) Hence show that the Jeffreys prior is

$$\pi(\mu, \sigma) \propto \frac{1}{\sigma^2}, \quad -\infty < \mu < \infty, \quad \sigma > 0.$$

Hint: For $i = 1, 2, \ldots, n$, $\log X_i \sim N(\mu, \sigma^2)$ and so $E(\log X_i) = \mu$ and $Var(\log X_i) = \sigma^2$.

[8 marks]

(d) Is this a proper prior distribution? Explain your answer.

[2 marks]

(e) Are μ and σ independent in this prior distribution? Explain your answer.

[2 marks]

[Total: 20 marks]

SECTION B

B3. A random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a normal $N(\mu, 1/\tau)$ distribution, with sample mean $\bar{x}$ and variance $s^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2/n$.

The likelihood function is

$$f(\boldsymbol{x}|\mu,\tau) = \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left[-\frac{n\tau}{2}\left\{s^2 + (\bar{x}-\mu)^2\right\}\right].$$

Suppose your prior beliefs about $(\mu, \tau)^T$ follow a normal-gamma $NGa(b, c, g, h)$ distribution.

(a) Show that your posterior distribution for $(\mu, \tau)^T$ is a $NGa(B, C, G, H)$ distribution where

$$B = \frac{bc + n\bar{x}}{c+n}, \quad C = c+n, \quad G = g+\frac{n}{2}, \quad H = h+\frac{cn(\bar{x}-b)^2}{2(c+n)}+\frac{ns^2}{2}.$$

Hint:

$$c(\mu - b)^2 + n(\bar{x}-\mu)^2 + 2h + ns^2 = C(\mu - B)^2 + 2H.$$

[**7 marks**]

(b) Use this joint posterior distribution to determine your marginal posterior density for μ. Name the posterior distribution for μ.

[**7 marks**]

(c) Suppose Y is the next value randomly sampled from the population. By writing $Y = \mu + \epsilon$, determine the predictive density for Y.

[**4 marks**]

[Please turn over for the rest of Question B3]

Question B3 continued on next page

(d) The aerobic capacity of 50 randomly chosen athletes was measured, giving a mean of $\bar{x} = 73.5$ ml/kg/min and standard deviation $s = 5.2$ ml/kg/min. These measurements are assumed to be a random sample from a Normal distribution with mean μ and precision τ.

Work with a sports scientist suggests the following prior distribution:

$$\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(80, 3, 2, 0.5).$$

(i) Obtain the joint posterior distribution for $(\mu, \tau)^T | \boldsymbol{x}$.

(ii) Find a 95% *highest density interval* for the mean aerobic capacity, μ.

(iii) Find a 95% prediction interval for another athlete to be accepted into the sample.

Hint: Note the following output from R:

```
> qt(0.975, 54)
[1] 2.004879
```

[**12 marks**]

[**Total: 30 marks**]

B4. Suppose you have a random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ from an Exponential-Gamma $EG(\alpha, \lambda)$ distribution, with likelihood function

$$f(\boldsymbol{x}|\alpha, \lambda) = \lambda^{n\alpha} \exp(\alpha n\bar{x} - \lambda n\bar{x}_{\text{exp}})/\Gamma(\alpha)^n,$$

where $\bar{x} = \sum_{i=1}^n x_i/n$ and $\bar{x}_{\text{exp}} = \sum_{i=1}^n \exp(x_i)/n$ are the sample means of x and $\exp(x)$.

(a) Assuming your prior distribution has $\alpha \sim Ga(a, b)$ and $\lambda \sim Ga(c, d)$, independently and for known values a, b, c and d, show that your posterior density is

$$\pi(\alpha, \lambda|\boldsymbol{x}) \propto \alpha^{a-1}\lambda^{n\alpha+c-1} \exp\left\{\alpha(n\bar{x} - b) - \lambda(n\bar{x}_{\text{exp}} + d)\right\}/\Gamma(\alpha)^n,$$

for $\alpha > 0$ and $\lambda > 0$.

[**5 marks**]

(b) Determine the posterior conditional densities for $\alpha|\lambda$ and $\lambda|\alpha$ up to a multiplicative constant.

[**8 marks**]

You decide to develop a Markov Chain Monte Carlo (MCMC) algorithm with separate steps for each parameter to simulate realisations from the posterior distribution. You have available a simulator which generates values from a gamma distribution.

(c) Why might a Metropolis within Gibbs algorithm be appropriate for simulating realisations from the posterior distribution?

[**3 marks**]

(d) You decide to use a normal $N(\alpha, \Sigma_\alpha)$ random walk proposal for α, that is, make proposals using $\alpha^*|\alpha \sim N(\alpha, \Sigma_\alpha)$. What is the acceptance probability for the proposed value α^*?

[**4 marks**]

(e) Write down the steps in your MCMC algorithm for simulating realisations from the posterior distribution.

[**10 marks**]

[**Total: 30 marks**]

Distributions for data

Binomial distribution

If $X|\theta \sim Bin(k,\theta)$ then it has probability function

$$f(x|\theta) = \binom{k}{x} \theta^x (1-\theta)^{k-x}, \quad x = 0, 1, \ldots, k,$$

where k is a positive integer and $0 < \theta < 1$. Also, $E(X) = k\theta$ and $Var(X) = k\theta(1-\theta)$.

Exponential distribution

If $X|\theta \sim Exp(\theta)$ then it has density

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0,$$

where $\theta > 0$. Also, $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$.

Exponential-Gamma distribution

If $X \sim EG(\alpha,\lambda)$ then it has density

$$f(x|\alpha,\lambda) = \lambda^\alpha \exp\left\{\alpha x - \lambda \exp(x)\right\}/\Gamma(\alpha),$$

where $\alpha > 0$ and $\lambda > 0$.

Gamma distribution

If $X|\alpha,\lambda \sim Ga(\alpha,\lambda)$ then it has density

$$f(x|\alpha,\lambda) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\alpha > 0$ and $\lambda > 0$. Also, $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$.

Log-normal distribution

If $X|\mu,\sigma \sim LN(\mu,\sigma^2)$ then it has density

$$f(x|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(\log x - \mu)^2\right\}, \quad x > 0$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Also, $E(X) = e^{\mu+\sigma^2/2}$, $Var(X) = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}$. Further $E(\log X) = \mu$ and $Var(\log X) = \sigma^2$.

Normal distribution

If $X|\mu, \tau \sim N(\mu, 1/\tau)$ then it has density

$$f(x|\mu,\tau) = \left(\frac{\tau}{2\pi}\right)^{1/2} \exp\left\{-\frac{\tau}{2}(x-\mu)^2\right\}, \quad x \in \mathbb{R}$$

where $\mu \in \mathbb{R}$ and $\tau > 0$. Also, $E(X) = \mu$ and $Var(X) = 1/\tau$. The distribution has the following quantiles

x	1.2816	1.6449	1.9600	2.3263	2.5758
$\Pr(X < x)$	0.9	0.95	0.975	0.99	0.995

Poisson distribution

If $X|\theta \sim Po(\theta)$ then it has probability function

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \ldots, .$$

where $\theta > 0$. Also, $E(X) = \theta$ and $Var(X) = \theta$.

Uniform distribution

If $X|\phi, \theta \sim U(\phi, \theta)$ then it has density

$$f(x|\phi,\theta) = \frac{1}{\theta - \phi}, \quad \phi < x < \theta,$$

where $\phi < \theta$. Also, $E(X) = (\phi + \theta)/2$ and $Var(X) = (\theta - \phi)^2/12$.

Distributions for prior beliefs

Beta distribution

If $\theta \sim Beta(g, h)$ then it has density

$$\pi(\theta) = \frac{\theta^{g-1}(1-\theta)^{h-1}}{B(g,h)}, \quad 0 < \theta < 1,$$

where $g > 0$ and $h > 0$. Also, $E(\theta) = g/(g+h)$ and $Var(\theta) = gh/\{(g+h)^2(g+h+1)\}$.

Gamma distribution

If $\theta \sim Ga(g, h)$ then it has density

$$\pi(\theta) = \frac{h^g \theta^{g-1} e^{-h\theta}}{\Gamma(g)}, \quad \theta > 0,$$

where $g > 0$ and $h > 0$. Also, $E(\theta) = g/h$ and $Var(\theta) = g/h^2$.

Generalised t distribution

If $\mu \sim t_a(b, c)$ then it has density

$$\pi(\mu) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{ac\pi}\,\Gamma\left(\frac{a}{2}\right)} \left\{1 + \frac{(\mu - b)^2}{ac}\right\}^{-\frac{a+1}{2}}, \quad \mu \in \mathbb{R},$$

where $b \in \mathbb{R}$, $a > 0$ and $c > 0$. Also, $E(\mu) = b$ and $Var(\mu) = ac/(a-2)$ if $a \geq 2$.

Inverse Chi distribution

If $\sigma \sim$ *Inv-Chi*(a, b) then it has density

$$\pi(\sigma|a,b) = \frac{2b^a \sigma^{-2a-1} e^{-b/\sigma^2}}{\Gamma(a)}, \quad \sigma > 0,$$

where $a > 0$, $b > 0$ and $\Gamma(a)$ is the gamma function. Also $E(\sigma) = \sqrt{b}\,\Gamma(a - 1/2)/\Gamma(a)$ and $Var(\sigma) = b/(a-1) - E(\sigma)^2$ if $a > 1$. The name of the distribution comes from the fact that $1/\sigma^2 \sim Ga(a, b) \equiv \chi^2_{2a}/(2b)$.

Log-normal distribution

If $\theta \sim LN(b, c^2)$ then it has density

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}\, c\, \theta} \exp\left\{-\frac{1}{2c^2}(\log\theta - b)^2\right\}, \quad \theta > 0$$

where $b \in \mathbb{R}$ and $c > 0$. Also, $E(\theta) = e^{b+c^2/2}$, $Var(\theta) = (e^{c^2} - 1)e^{2b+c^2}$. Further $\log\theta \sim N(b, c^2)$ and so $E(\log\theta) = b$ and $Var(\log\theta) = c^2$.

Normal distribution

If $\mu \sim N(b, 1/d)$ then it has density

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\}, \quad \mu \in \mathbb{R},$$

where $b \in \mathbb{R}$ and $c > 0$. Also, $E(\mu) = b$ and $Var(\mu) = 1/d$.

Normal-gamma distribution

If $\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(b, c, g, h)$ then it has density

$$\pi(\mu, \tau) \propto \tau^{g-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[c(\mu - b)^2 + 2h\right]\right\}, \quad \mu \in \mathbb{R},\ \tau > 0$$

where $b \in \mathbb{R}$ and $c, g, h > 0$. Also, $\mu|\tau \sim N\left(b, \dfrac{1}{c\tau}\right)$, $\tau \sim Ga(g, h)$ and has marginal distribution $\mu \sim t_{2g}\left(b, \dfrac{h}{gc}\right)$.

Uniform distribution

If $\theta \sim U(a, b)$ then it has density

$$\pi(\theta) = \frac{1}{b - a}, \quad a < \theta < b,$$

where $a < b$. Also, $E(\theta) = (a + b)/2$ and $Var(\theta) = (b - a)^2/12$.

THE END