NEWCASTLE UNIVERSITY

SCHOOL OF MATHEMATICS, STATISTICS & PHYSICS

SEMESTER 1 2019/20

MAS3902: Specimen Paper 2

Bayesian Inference

Time allowed: 2 hours

Candidates should attempt all questions. Marks for each question are indicated. However you are advised that marks may be adjusted in accordance with the University's Moderation and Scaling Policy.

There are TWO questions in Section A and TWO questions in Section B.

Calculators may be used.

SECTION A

A1. Suppose that the lifetimes X of n patients randomly allocated to receive a new drug treatment follow a $Ga(4, \theta)$ distribution.

(a) Show that the likelihood function for θ given the lifetimes $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ of a random sample of patients given the treatment is

$$f(\boldsymbol{x}|\theta) \propto \theta^{4n} e^{-n\bar{x}\theta}, \quad \theta > 0,$$

where $\bar{x}$ is the mean lifetime in the sample.

[4 marks]

(b) Suppose your prior beliefs about θ were described by a $Ga(g, h)$ distribution. Determine your posterior density for θ. Name this distribution, including its parameters.

[5 marks]

(c) Is the *Gamma* distribution a conjugate prior distribution in this case? Explain your answer.

[2 marks]

(d) Explain why, in general, a 95% highest density interval (HDI) is not straightforward to calculate for this posterior distribution.

[3 marks]

(e) Determine an approximate 95% HDI for θ by approximating the posterior distribution by a normal distribution with the same posterior mean and variance.

[6 marks]

[Total: 20 marks]

A2. The time to fracture, X, in metals used in aeroplane fuselages that are subject to the growth of fatigue cracks is modelled by a reparameterised log-normal $RLN(\mu, \alpha)$ distribution, where $(\mu, \alpha) \in \mathbb{R}^2$.

(a) Show that the likelihood function is

$$f(\boldsymbol{x}|\mu, \alpha) \propto e^{n\alpha/2} \exp\left\{-\frac{e^\alpha}{2}\sum_{i=1}^{n}(\log x_i - \mu)^2\right\}.$$

[**4 marks**]

(b) Show that the Jeffreys prior is, for $(\mu, \alpha) \in \mathbb{R}^2$,

$$\pi(\mu, \alpha) \propto e^{\alpha/2}.$$

Hint: If $X \sim RLN(\mu, \alpha)$, then $E(\log X) = \mu$ and $Var(\log X) = e^{-\alpha}$.

[**14 marks**]

(c) Are μ and σ independent in this prior distribution? Explain your answer.

[**2 marks**]

[**Total: 20 marks**]

SECTION B

B3. A random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ is taken from a gamma $Ga(\alpha, \lambda)$, and has sample mean $\bar{x}$. Suppose the prior belief about λ follows a mixture gamma distribution

$$\lambda \sim p_1 Ga(g_1, g_1) + p_2 Ga(g_2, g_2).$$

(a) Determine the mean and variance of the prior distribution, and show that the prior mean does not depend on p_i and g_i, $i = 1, 2$.

[11 marks]

(b) Show that the posterior distribution is of the form

$$p_1^* Ga(G_1, H_1) + p_2^* Ga(G_2, H_2),$$

and show that $G_i = n\alpha + g_i$ and $H_i = g_i + n\bar{x}$, $i = 1, 2$.

[11 marks]

(c) Find expressions for p_1^* and p_2^*.

[8 marks]

[Total: 30 marks]

B4. Suppose you have a random sample $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ from a reparameterised inverse Gaussian $RIG(\mu, \lambda)$ distribution, where μ and λ are unknown. Your prior distribution has $\mu \sim Exp(a)$ and $\lambda \sim Ga(g, h)$, independently, for known values a, g and h.

(a) Show that the likelihood function is

$$f(\boldsymbol{x}|\mu, \lambda) \propto \lambda^{n/2} \exp\left\{-\frac{n\lambda}{2\mu^2}\left(\bar{x} - 2\mu + \frac{\mu^2}{\bar{x}_h}\right)\right\},$$

where $\bar{x}$ is the arithmetic mean of the data and $\bar{x}_h$ is the harmonic mean of the data, given by

$$\frac{1}{\bar{x}_h} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}.$$

[**5 marks**]

(b) Determine your posterior density for $(\mu, \lambda)^T$.

[**4 marks**]

(c) Determine the conditional posterior densities for $\mu|\lambda$ and $\lambda|\mu$ up to a multiplicative constant.

[**8 marks**]

You decide to develop an MCMC algorithm with separate steps for each parameter to simulate realisations from the posterior distribution.

(d) Why do you need to use a Metropolis within Gibbs algorithm?

[**2 marks**]

(e) You decide to use a symmetric normal random walk proposal for μ with known variance Σ_μ, that is, $q(\mu^*|\mu) = N(\mu, \Sigma_\mu)$. What is the acceptance probability for the proposed value μ^*?

[**4 marks**]

(f) Explain how you might initialise your algorithm.

[**1 mark**]

(g) Write down the steps in your MCMC algorithm for simulating realisations from the posterior distribution.

[**6 marks**]

[**Total: 30 marks**]

Distributions for data

Binomial distribution

If $X|\theta \sim Bin(k,\theta)$ then it has probability function

$$f(x|\theta) = \binom{k}{x}\theta^x(1-\theta)^{k-x}, \quad x = 0,1,\ldots,k,$$

where k is a positive integer and $0 < \theta < 1$. Also, $E(X) = k\theta$ and $Var(X) = k\theta(1-\theta)$.

Exponential distribution

If $X|\theta \sim Exp(\theta)$ then it has density

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0,$$

where $\theta > 0$. Also, $E(X) = 1/\lambda$ and $Var(X) = 1/\lambda^2$.

Gamma distribution

If $X|\alpha,\lambda \sim Ga(\alpha,\lambda)$ then it has density

$$f(x|\alpha,\lambda) = \frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\alpha > 0$ and $\lambda > 0$. Also, $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$.

Normal distribution

If $X|\mu,\tau \sim N(\mu,1/\tau)$ then it has density

$$f(x|\mu,\tau) = \left(\frac{\tau}{2\pi}\right)^{1/2}\exp\left\{-\frac{\tau}{2}(x-\mu)^2\right\}, \quad x \in \mathbb{R}$$

where $\mu \in \mathbb{R}$ and $\tau > 0$. Also, $E(X) = \mu$ and $Var(X) = 1/\tau$. The distribution has the following quantiles

x	1.2816	1.6449	1.9600	2.3263	2.5758
$\Pr(X < x)$	0.9	0.95	0.975	0.99	0.995

Poisson distribution

If $X|\theta \sim Po(\theta)$ then it has probability function

$$f(x|\theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, \ldots, .$$

where $\theta > 0$. Also, $E(X) = \theta$ and $Var(X) = \theta$.

Reparameterised inverse Gaussian distribution

If $X|\mu, \lambda \sim RIG(\mu, \lambda)$ then it has density

$$f(x|\mu, \lambda) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left\{-\frac{\lambda}{2x\mu^2}(x-\mu)^2\right\}, \quad x > 0, \quad \mu > 0, \lambda > 0.$$

Also, $E(X) = \mu$ and $Var(X) = \mu^3/\lambda$.

Reparameterised log-normal distribution

If $X|\mu, \alpha \sim RLN(\mu, \alpha)$ then it has density

$$f(x|\mu, \alpha) = \frac{e^{\alpha/2}}{x\sqrt{2\pi}} \exp\left\{-\frac{e^{\alpha}}{2}(\log x - \mu)^2\right\}, \quad x > 0,$$

where $-\infty < \mu < \infty$ and $-\infty < \alpha < \infty$. Also, $E(X) = \exp\{\mu + e^{-\alpha}/2\}$, $Var(X) = \{\exp(e^{-\alpha}) - 1\}\exp(2\mu + e^{-\alpha})$. Further, $E(\log X) = \mu$ and $Var(\log X) = e^{-\alpha}$.

Uniform distribution

If $X|\phi, \theta \sim U(\phi, \theta)$ then it has density

$$f(x|\phi, \theta) = \frac{1}{\theta - \phi}, \quad \phi < x < \theta,$$

where $\phi < \theta$. Also, $E(X) = (\phi + \theta)/2$ and $Var(X) = (\theta - \phi)^2/12$.

Distributions for prior beliefs

Beta distribution

If $\theta \sim Beta(g, h)$ then it has density

$$\pi(\theta) = \frac{\theta^{g-1}(1-\theta)^{h-1}}{B(g,h)}, \quad 0 < \theta < 1,$$

where $g > 0$ and $h > 0$. Also, $E(\theta) = g/(g+h)$ and $Var(\theta) = gh/\{(g+h)^2(g+h+1)\}$.

Gamma distribution

If $\theta \sim Ga(g, h)$ then it has density

$$\pi(\theta) = \frac{h^g \theta^{g-1} e^{-h\theta}}{\Gamma(g)}, \quad \theta > 0,$$

where $g > 0$ and $h > 0$. Also, $E(\theta) = g/h$ and $Var(\theta) = g/h^2$.

Generalised *t* distribution

If $\mu \sim t_a(b, c)$ then it has density

$$\pi(\mu) = \frac{\Gamma\left(\frac{a+1}{2}\right)}{\sqrt{ac\pi}\,\Gamma\left(\frac{a}{2}\right)} \left\{1 + \frac{(\mu-b)^2}{ac}\right\}^{-\frac{a+1}{2}}, \quad \mu \in \mathbb{R},$$

where $b \in \mathbb{R}$, $a > 0$ and $c > 0$. Also, $E(\mu) = b$ and $Var(\mu) = ac/(a-2)$ if $a \geq 2$.

Inverse Chi distribution

If $\sigma \sim$ *Inv-Chi*(a, b) then it has density

$$\pi(\sigma|a,b) = \frac{2b^a \sigma^{-2a-1} e^{-b/\sigma^2}}{\Gamma(a)}, \quad \sigma > 0,$$

where $a > 0$, $b > 0$ and $\Gamma(a)$ is the gamma function. Also $E(\sigma) = \sqrt{b}\,\Gamma(a-1/2)/\Gamma(a)$ and $Var(\sigma) = b/(a-1) - E(\sigma)^2$ if $a > 1$. The name of the distribution comes from the fact that $1/\sigma^2 \sim Ga(a, b) \equiv \chi^2_{2a}/(2b)$.

Log-normal distribution

If $\theta \sim LN(b, c^2)$ then it has density

$$\pi(\theta) = \frac{1}{\sqrt{2\pi}\, c\, \theta} \exp\left\{-\frac{1}{2c^2}(\log\theta - b)^2\right\}, \quad \theta > 0$$

where $b \in \mathbb{R}$ and $c > 0$. Also, $E(\theta) = e^{b+c^2/2}$, $Var(\theta) = (e^{c^2} - 1)e^{2b+c^2}$. Further $\log\theta \sim N(b, c^2)$ and so $E(\log\theta) = b$ and $Var(\log\theta) = c^2$.

Normal distribution

If $\mu \sim N(b, 1/d)$ then it has density

$$\pi(\mu) = \left(\frac{d}{2\pi}\right)^{1/2} \exp\left\{-\frac{d}{2}(\mu - b)^2\right\}, \quad \mu \in \mathbb{R},$$

where $b \in \mathbb{R}$ and $c > 0$. Also, $E(\mu) = b$ and $Var(\mu) = 1/d$.

Normal-gamma distribution

If $\begin{pmatrix} \mu \\ \tau \end{pmatrix} \sim NGa(b, c, g, h)$ then it has density

$$\pi(\mu, \tau) \propto \tau^{g-\frac{1}{2}} \exp\left\{-\frac{\tau}{2}\left[c(\mu - b)^2 + 2h\right]\right\}, \quad \mu \in \mathbb{R},\ \tau > 0$$

where $b \in \mathbb{R}$ and $c, g, h > 0$. Also, $\mu|\tau \sim N\left(b, \frac{1}{c\tau}\right)$, $\tau \sim Ga(g, h)$ and has marginal distribution $\mu \sim t_{2g}\left(b, \frac{h}{gc}\right)$.

Uniform distribution

If $\theta \sim U(a, b)$ then it has density

$$\pi(\theta) = \frac{1}{b - a}, \quad a < \theta < b,$$

where $a < b$. Also, $E(\theta) = (a + b)/2$ and $Var(\theta) = (b - a)^2/12$.

THE END