MAS2903
Problems Class 2

Dr. Lee Fawcett

Semester 2, 2019—2020

Mid-semester test feedback

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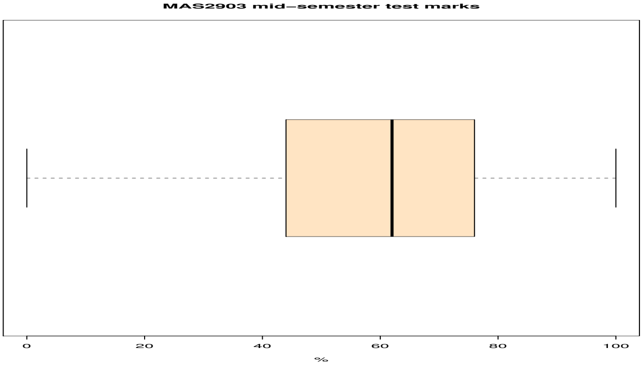
- Not sure you were expecting *MATCH* output (although I did say you should revise up to page 66)
- Expect *MATCH* output in the exam!
- Official marks in NESS have been adjusted slightly to account for overly-harsh marking – done fairly, across the board

Mid-semester test feedback

Q1	Median	Mean	Q3	Max.	St. dev.
44	62	58	76	100	26



Specific feedback: Question 1 (3 marks)

- Generally very well done

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- Some people not reading carefully – just assuming that the spinner was classical, and that the jockey was a Bayesian...
- Some students choosing a different word for each scenario

Specific feedback: Question 2 (8 marks)

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- Language: “Speed cameras = 16/47" (as opposed to $\Pr(SC|R) = 16/47$)

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- Comments at end – done quite well, by those who attempted it. Should also comment on prior/data discrepancies

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  - Also, pulling the 3 out of the product and not raising it to the power n
- Part (b)
  - Writing $T = \sum_{i=1}^{n} x_i^3$ – i.e. lower case x
  - Generally, just errors in notation

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- Not having square root over $d + n\tau$
- Incorrectly identifying τ and d
- Mistakes getting inequalities the wrong way round
- General arithmetic mistakes

Assignment feedback

- **Question 4** (Air France disaster)
  - Done well by everyone
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- **Question 4** (Air France disaster)
  - Done well by everyone
  - Some issues with "odds" – see car example in Chapter 1!
- **Question 9** (Rayleigh likelihood and sufficient statistic)
  - Main problem – students not choosing the correct functions h and g and ending up with two sufficient statistics... only one parameter!
  - Otherwise done well

Assignment feedback

- **Question 10** (Henry Cavendish on the Earth's density)
  - Generally done well
  - A lot of

$$\Pr(5.45 < \mu < 5.55) = \Pr(\mu < 5.55) - \Pr(\mu > 5.45)$$

- **Question 11** (Posterior $\propto$ Prior $\times$ Likelihood)
  - Exponential prior – PDF should not contain *x*! We have $\theta \sim \text{Exp}(1)$, so

$$\pi(\theta) = 1 \cdot e^{-1\cdot\theta} = e^{-\theta}.$$

  - Confusion over the binomial range

Assignment feedback

- **Question 12** (Bicycle tyre punctures)
  - Main comment – students bnot commenting on the change from prior to posterior!

Chapter 5: Question 29

Using a random sample from a $Bin(k, \theta)$ (with k known), determine the posterior distribution for θ assuming

(i) vague prior knowledge;

(ii) the Jeffreys prior distribution;

(iii) a very large sample.

Solution

The **conjugate prior distribution** is a *Beta*(g, h) distribution. Using this prior distribution, the posterior density is

$$\begin{aligned}\pi(\theta|\boldsymbol{x}) &\propto \pi(\theta)\, f(\boldsymbol{x}|\theta) \\ &\propto \theta^{g-1}(1-\theta)^{h-1} \times \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{k-x_i}, \qquad 0 < \theta < 1\end{aligned}$$

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&\propto \theta^{g+n\bar{x}-1}(1-\theta)^{h+nk-n\bar{x}-1}, \qquad 0<\theta<1
\end{aligned}
$$

$$i.e.\ \theta|\boldsymbol{x} \sim Beta(G = g + n\bar{x}, H = h + nk - n\bar{x}).$$

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We represent **vague prior information** by taking a conjugate prior distribution with large variance.

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Thus we will take this limit to represent vague prior information.Hence the posterior distribution under vague prior information is

$$\theta|\boldsymbol{x} \sim Beta(n\bar{x}, nk - n\bar{x}).$$

Solution

The **Jeffreys prior distribution** is

$$\pi(\theta) \propto \sqrt{I(\theta)}.$$

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Now

$$\begin{aligned} f(\boldsymbol{x}|\theta) &\propto \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{k-x_i} \\ &\propto \theta^{n\bar{x}}(1-\theta)^{kn-n\bar{x}}. \end{aligned}$$

Solution

Therefore

$$\log f(\boldsymbol{x}|\theta) \quad = \quad constant + n\bar{x}\log\theta + n(k-\bar{x})\log(1-\theta)$$

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$$\begin{aligned}\log f(\boldsymbol{x}|\theta) &= constant + n\bar{x}\log\theta + n(k-\bar{x})\log(1-\theta) \\ &\Rightarrow \quad \frac{\partial}{\partial\theta}\log f(\boldsymbol{x}|\theta) = \frac{n\bar{x}}{\theta} - \frac{n(k-\bar{x})}{1-\theta}\end{aligned}$$

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Therefore

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\log f(\boldsymbol{x}|\theta) &= constant + n\bar{x}\log\theta + n(k-\bar{x})\log(1-\theta) \\
&\Rightarrow \quad \frac{\partial}{\partial\theta}\log f(\boldsymbol{x}|\theta) = \frac{n\bar{x}}{\theta} - \frac{n(k-\bar{x})}{1-\theta} \\
&\Rightarrow \quad \frac{\partial^2}{\partial\theta^2}\log f(\boldsymbol{x}|\theta) = -\frac{n\bar{x}}{\theta^2} - \frac{n(k-\bar{x})}{(1-\theta)^2}
\end{aligned}$$

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$$\Rightarrow \quad I(\theta) = E_{\boldsymbol{X}|\theta}\left[-\frac{\partial^2}{\partial\theta^2}\log f(\boldsymbol{X}|\theta)\right]$$

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&\qquad = \frac{nE_{\boldsymbol{X}|\theta}(\bar{X})}{\theta^2} + \frac{n[k-E_{\boldsymbol{X}|\theta}(\bar{X})]}{(1-\theta)^2}.
\end{aligned}
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Now $E_{\boldsymbol{X}|\theta}(\bar{X}) = E_{X|\theta}(X) = k\theta$. Therefore

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Hence the Jeffreys prior for this model is

$$\begin{aligned}\pi(\theta) &\propto \sqrt{I(\theta)} \\ &\propto \sqrt{\frac{nk}{\theta(1-\theta)}}, \qquad 0 < \theta < 1\end{aligned}$$

Solution

Now $E_{\boldsymbol{X}|\theta}(\bar{X}) = E_{X|\theta}(X) = k\theta$. Therefore

$$I(\theta) = \frac{nk}{\theta} + \frac{nk}{1-\theta} = \frac{nk}{\theta(1-\theta)}.$$

Hence the Jeffreys prior for this model is

$$\begin{aligned}
\pi(\theta) &\propto \sqrt{I(\theta)} \\
&\propto \sqrt{\frac{nk}{\theta(1-\theta)}}, \qquad 0 < \theta < 1 \\
&\propto \theta^{-1/2}(1-\theta)^{-1/2}, \qquad 0 < \theta < 1.
\end{aligned}$$

Solution

This is a *Beta*$(1/2, 1/2)$ prior distribution and so the resulting posterior distribution is

$$\theta|\boldsymbol{x} \sim Beta(1/2 + n\bar{x}, 1/2 + nk - n\bar{x}).$$

Solution

The **asymptotic posterior distribution** (as $n \to \infty$) is

$$\theta|\boldsymbol{x} \sim N\left(\hat{\theta}, J(\hat{\theta})^{-1}\right),$$

where

$$J(\theta) = -\frac{\partial^2}{\partial\theta^2}\log f(\boldsymbol{x}|\theta) = \frac{n\bar{x}}{\theta^2} + \frac{n(k-\bar{x})}{(1-\theta)^2}.$$

Solution

Now

$$\frac{\partial}{\partial\theta}\log f(\boldsymbol{x}|\theta) = 0 \implies \frac{n\bar{x}}{\hat{\theta}} - \frac{n(k-\bar{x})}{1-\hat{\theta}} = 0$$

$$\implies \hat{\theta} = \frac{\bar{x}}{k}$$

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$$\implies J(\hat{\theta}) = \frac{nk^3}{\bar{x}(k-\bar{x})}$$

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\frac{\partial}{\partial\theta}\log f(\boldsymbol{x}|\theta) = 0 \quad &\Longrightarrow \quad \frac{n\bar{x}}{\hat{\theta}} - \frac{n(k-\bar{x})}{1-\hat{\theta}} = 0 \\
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&\Longrightarrow \quad J(\hat{\theta}) = \frac{nk^3}{\bar{x}(k-\bar{x})} \\
&\Longrightarrow \quad J(\hat{\theta})^{-1} = \frac{\bar{x}(k-\bar{x})}{nk^3}.
\end{aligned}$$

Solution

Therefore, for large n, the posterior distribution for θ is

$$\theta|\boldsymbol{x} \sim N\left(\frac{\bar{x}}{k}, \frac{\bar{x}(k-\bar{x})}{nk^3}\right) \qquad \text{approximately.}$$