MAS2903
Problems Class 1

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Semester 2, 2019—2020

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- Aim of problems classes: to work through some of the questions in Chapter 5

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  - Due in by **3pm, Friday 6th March**
  - Consists of questions 4, 9, 10, 11, 12
  - You might want to wait until after next week's lectures before attempting question 10

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  - Consists of questions 4, 9, 10, 11, 12
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- Next week: drop-in – come along with any questions you might have about the assignment
- Next week: computer practical on Tuesday at 12pm, to try out some of the R questions in Chapter 5

Question 5

An experiment can result in four possible outcomes $x_1,\ x_2,\ x_3$ and x_4.

The probability of each of these outcomes is affected by a parameter θ which can take one of three values: θ_1, θ_2 and θ_3:

$p(x\|\theta)$	x_1	x_2	x_3	x_4
θ_1	0.1	0.2	0.3	0.4
θ_2	0.3	0.3	0.2	0.2
θ_3	0.5	0.1	0.1	0.3

Question 5

The prior plausibilities attaching to θ_1, θ_2 and θ_3 are 0.1, 0.4 and 0.5.

Find the posterior probability function for θ after observing x_1. Repeat the calculations for x_2, x_3 and x_4.

Solution to Question 5

The **posterior probabilities** given x_1 are

$$Pr(\theta_1|x_1) = \frac{Pr(x_1|\theta_1)Pr(\theta_1)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)}$$

Solution to Question 5

The **posterior probabilities** given x_1 are

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$$= \frac{0.1 \times 0.1}{0.1 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.5}$$

Solution to Question 5

The **posterior probabilities** given x_1 are

$$\begin{aligned}
Pr(\theta_1|x_1) &= \frac{Pr(x_1|\theta_1)Pr(\theta_1)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)} \\
&= \frac{0.1 \times 0.1}{0.1 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.5} \\
&= 0.02632,
\end{aligned}$$

Solution to Question 5

The **posterior probabilities** given x_1 are

$$
\begin{aligned}
Pr(\theta_1|x_1) &= \frac{Pr(x_1|\theta_1)Pr(\theta_1)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)} \\
&= \frac{0.1 \times 0.1}{0.1 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.5} \\
&= 0.02632,
\end{aligned}
$$

$$
Pr(\theta_2|x_1) = \frac{Pr(x_1|\theta_2)Pr(\theta_2)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)}
$$

Solution to Question 5

The **posterior probabilities** given x_1 are

$$\begin{aligned}
Pr(\theta_1|x_1) &= \frac{Pr(x_1|\theta_1)Pr(\theta_1)}{Pr(x_1|\theta_1)Pr(\theta_1)+Pr(x_1|\theta_2)Pr(\theta_2)+Pr(x_1|\theta_3)Pr(\theta_3)} \\
&= \frac{0.1\times 0.1}{0.1\times 0.1+0.3\times 0.4+0.5\times 0.5} \\
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&= \frac{0.3\times 0.4}{0.1\times 0.1+0.3\times 0.4+0.5\times 0.5}
\end{aligned}$$

Solution to Question 5

The **posterior probabilities** given x_1 are

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\begin{aligned}
Pr(\theta_1|x_1) &= \frac{Pr(x_1|\theta_1)Pr(\theta_1)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)} \\
&= \frac{0.1 \times 0.1}{0.1 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.5} \\
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Pr(\theta_2|x_1) &= \frac{Pr(x_1|\theta_2)Pr(\theta_2)}{Pr(x_1|\theta_1)Pr(\theta_1) + Pr(x_1|\theta_2)Pr(\theta_2) + Pr(x_1|\theta_3)Pr(\theta_3)} \\
&= \frac{0.3 \times 0.4}{0.1 \times 0.1 + 0.3 \times 0.4 + 0.5 \times 0.5} \\
&= 0.31579,
\end{aligned}
$$

Solution to Question 5

and

$$Pr(\theta_3|x_1) = 1 - Pr(\theta_1|x_1) - Pr(\theta_2|x_1)$$

Solution to Question 5

and

$$
\begin{aligned}
Pr(\theta_3|x_1) &= 1 - Pr(\theta_1|x_1) - Pr(\theta_2|x_1) \\
&= 1 - 0.02632 - 0.31579
\end{aligned}
$$

Solution to Question 5

and

$$\begin{aligned} Pr(\theta_3|x_1) &= 1 - Pr(\theta_1|x_1) - Pr(\theta_2|x_1) \\ &= 1 - 0.02632 - 0.31579 \\ &= 0.65789. \end{aligned}$$

Thus, after observing x_1, we have the following probability function for θ:

and

$$\begin{aligned} Pr(\theta_3|x_1) &= 1 - Pr(\theta_1|x_1) - Pr(\theta_2|x_1) \\ &= 1 - 0.02632 - 0.31579 \\ &= 0.65789. \end{aligned}$$

Thus, after observing x_1, we have the following probability function for θ:

θ	θ_1	θ_2	θ_3
$\Pr(\theta = \theta_i \mid x_1)$	0.02632	0.31579	0.65789

Same for x_2, x_3 and x_4.

Question 6

Extensive manuscripts of two scribes show that they differ on the relative frequencies with which they use two alternative vowel forms *e* and *œ*, as in the following table:

	e	*œ*
Scribe A	0.3	0.7
Scribe B	0.5	0.5

Question 6

A new manuscript fragment, known to be by one of the scribes, uses the vowel form five times, three times as *e* and twice as *œ*.

If an independent assessment of similar fragments favours Scribe A as author with probability 0.7, show that the evidence from the vowel form is sufficient to turn the odds in favour of Scribe B.

Solution to Question 6

Let X = number of e's observed.

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Also, let θ_A denote Scribe A and θ_B denote Scribe B.

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Also, let θ_A denote Scribe A and θ_B denote Scribe B.

Then

$$Pr(\theta_A) = 0.7, \quad Pr(\theta_B) = 0.3,$$

$$X|\theta_A \sim Bin(5, 0.3) \quad \text{and} \quad X|\theta_B \sim Bin(5, 0.5).$$

Solution to Question 6

Hence

$$Pr(X = 3|\theta_A) = \binom{5}{3} 0.3^3 \times 0.7^2 = 0.1323$$

Solution to Question 6

Hence

$$
\begin{aligned}
Pr(X = 3|\theta_A) &= \binom{5}{3} 0.3^3 \times 0.7^2 = 0.1323 \\
Pr(X = 3|\theta_B) &= \binom{5}{3} 0.5^3 \times 0.5^2 = 0.3125.
\end{aligned}
$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$Pr(\theta_A|x=3) = \frac{Pr(X=3|\theta_A)Pr(\theta_A)}{Pr(X=3|\theta_A)Pr(\theta_A) + Pr(X=3|\theta_B)Pr(\theta_B)}$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$Pr(\theta_A|x=3) = \frac{Pr(X=3|\theta_A)Pr(\theta_A)}{Pr(X=3|\theta_A)Pr(\theta_A)+Pr(X=3|\theta_B)Pr(\theta_B)}$$

$$= \frac{0.1323\times 0.7}{0.1323\times 0.7+0.3125\times 0.3}$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$
\begin{aligned}
Pr(\theta_A | x = 3) &= \frac{Pr(X = 3|\theta_A)Pr(\theta_A)}{Pr(X = 3|\theta_A)Pr(\theta_A) + Pr(X = 3|\theta_B)Pr(\theta_B)} \\
&= \frac{0.1323 \times 0.7}{0.1323 \times 0.7 + 0.3125 \times 0.3} \\
&= 0.497 \;\; (3\, d.p.)
\end{aligned}
$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$\begin{aligned} Pr(\theta_A|x=3) &= \frac{Pr(X=3|\theta_A)Pr(\theta_A)}{Pr(X=3|\theta_A)Pr(\theta_A)+Pr(X=3|\theta_B)Pr(\theta_B)} \\ &= \frac{0.1323\times 0.7}{0.1323\times 0.7+0.3125\times 0.3} \\ &= 0.497 \;\; (3\,d.p.) \end{aligned}$$

$$Pr(\theta_B|x=3) = 1-Pr(\theta_A|x=3) = 0.503 \;\; (3\,d.p.).$$

Solution to Question 6

Therefore, using Bayes' Theorem

$$Pr(\theta_A|x=3) = \frac{Pr(X=3|\theta_A)Pr(\theta_A)}{Pr(X=3|\theta_A)Pr(\theta_A)+Pr(X=3|\theta_B)Pr(\theta_B)}$$

$$= \frac{0.1323\times 0.7}{0.1323\times 0.7+0.3125\times 0.3}$$

$$= 0.497 \;\; (3\,d.p.)$$

$$Pr(\theta_B|x=3) = 1-Pr(\theta_A|x=3) = 0.503 \;\; (3\,d.p.).$$

Thus, the odds have turned from 7 : 3 in favour of Scribe A to 503 : 497 in favour of Scribe B.

Question 7

If $X_1, X_2, \ldots, X_n$ are independent random variables with probability function

$$f(x|\theta) = \theta(1-\theta)^{x-1}, \quad x = 1, 2, \ldots,$$

show that $T = \sum_{i=1}^{n} X_i$ is sufficient for θ.

Solution to Question 7

The joint probability function is

$$f(\boldsymbol{x}|\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i - 1}$$

Solution to Question 7

The joint probability function is

$$\begin{aligned} f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} \\ &= \theta^n(1-\theta)^{x_1-1}(1-\theta)^{x_2-1} \times \ldots \times (1-\theta)^{x_n-1} \end{aligned}$$

Solution to Question 7

The joint probability function is

$$\begin{aligned} f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} \\ &= \theta^n(1-\theta)^{x_1-1}(1-\theta)^{x_2-1} \times \ldots \times (1-\theta)^{x_n-1} \\ &= \theta^n(1-\theta)^{\sum x_i - n} \end{aligned}$$

Solution to Question 7

The joint probability function is

$$\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} \\
&= \theta^n (1-\theta)^{x_1-1}(1-\theta)^{x_2-1} \times \ldots \times (1-\theta)^{x_n-1} \\
&= \theta^n (1-\theta)^{\sum x_i - n} \\
&= 1 \times \theta^n (1-\theta)^{\sum x_i - n}
\end{aligned}$$

Solution to Question 7

The joint probability function is

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f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} \\
&= \theta^n (1-\theta)^{x_1-1}(1-\theta)^{x_2-1} \times \ldots \times (1-\theta)^{x_n-1} \\
&= \theta^n (1-\theta)^{\sum x_i - n} \\
&= 1 \times \theta^n (1-\theta)^{\sum x_i - n} \\
&= h(\boldsymbol{x})\, g\left(\sum x_i, \theta\right)
\end{aligned}
$$

Solution to Question 7

The joint probability function is

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\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} \\
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&= h(\boldsymbol{x})\, g\left(\sum x_i, \theta\right)
\end{aligned}
$$

where $h(\boldsymbol{x}) = 1$ and $g(t,\theta) = \theta^n(1-\theta)^{t-n}$. Therefore, by the Factorisation Theorem, $T = \sum X_i$ is sufficient for θ.

Question 8

If $X_1, X_2, \ldots, X_n$ are independent $N(5, \sigma^2)$ random variables, show that

$$T = \sum_{i=1}^{n} (X_i - 5)^2$$

is sufficient for σ.

Solution to Question 8

The joint probability density function is

$$f(\boldsymbol{x}|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i - 5)^2}{2\sigma^2}\right\}$$

Solution to Question 8

The joint probability density function is

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\begin{aligned}
f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i-5)^2}{2\sigma^2}\right\} \\
&= (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-5)^2\right\}
\end{aligned}
$$

Solution to Question 8

The joint probability density function is

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&= 1 \times (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-5)^2\right\}
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Solution to Question 8

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f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i-5)^2}{2\sigma^2}\right\} \\
&= (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-5)^2\right\} \\
&= 1 \times (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-5)^2\right\} \\
&= h(\boldsymbol{x})\, g\left(\sum (x_i-5)^2, \sigma\right)
\end{aligned}$$

Solution to Question 8

The joint probability density function is

$$\begin{aligned} f(\boldsymbol{x}|\theta) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_i-5)^2}{2\sigma^2}\right\} \\ &= (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-5)^2\right\} \\ &= 1 \times (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-5)^2\right\} \\ &= h(\boldsymbol{x})\, g\left(\sum (x_i-5)^2, \sigma\right) \end{aligned}$$

where $h(\boldsymbol{x}) = 1$ and $g(t,\sigma) = (2\pi)^{-n/2}\sigma^{-n}e^{-t/(2\sigma^2)}$.

Solution to Question 8

The joint probability density function is

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&= 1 \times (2\pi)^{-n/2}\sigma^{-n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-5)^2\right\} \\
&= h(\boldsymbol{x})\, g\left(\sum (x_i-5)^2, \sigma\right)
\end{aligned}
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where $h(\boldsymbol{x}) = 1$ and $g(t, \sigma) = (2\pi)^{-n/2}\sigma^{-n}e^{-t/(2\sigma^2)}$.

Therefore, by the Factorisation Theorem, $T = \sum (X_i - 5)^2$ is sufficient for σ.