Lecture 2: Inverse CDF method

Today's lecture

In this lecture we look at the **inverse CDF method** for simulating from continuous random variables.

The inverse CDF method

This is a method for simulating univariate continuous random variables

- Let $U \sim U(0, 1)$
- Suppose $F(x)$ is a well-defined CDF which is invertible
- Then the random variable $X = F^{-1}(U)$ has CDF $F(x)$

The inverse CDF method

$$X = F^{-1}(U).$$

So

$$\begin{aligned} \Pr(X \le x) &= \Pr\left(F^{-1}(U) \le x\right) \\ &= \Pr\left(F(F^{-1}(U)) \le F(x)\right) \\ &= \Pr(U \le F(x)) \\ &= F(x), \end{aligned}$$

since $0 \le F(x) \le 1$ (see sketch on board).

The inverse CDF method

$$X = F^{-1}(U).$$

So

$$\begin{aligned}
\Pr(X \le x) &= \Pr\left(F^{-1}(U) \le x\right) \\
&= \Pr\left(F(F^{-1}(U)) \le F(x)\right) \\
&= \Pr(U \le F(x)) \\
&= F(x),
\end{aligned}$$

since $0 \le F(x) \le 1$ (see sketch on board).

The inverse CDF method

$$X = F^{-1}(U).$$

So

$$\begin{aligned}
\Pr(X \leq x) &= \Pr\left(F^{-1}(U) \leq x\right) \\
&= \Pr\left(F(F^{-1}(U)) \leq F(x)\right) \\
&= \Pr(U \leq F(x)) \\
&= F(x),
\end{aligned}$$

since $0 \leq F(x) \leq 1$ (see sketch on board).

The inverse CDF method

$$X = F^{-1}(U).$$

So

$$\begin{aligned}
\Pr(X \leq x) &= \Pr\big(F^{-1}(U) \leq x\big) \\
&= \Pr\big(F(F^{-1}(U)) \leq F(x)\big) \\
&= \Pr(U \leq F(x)) \\
&= F(x),
\end{aligned}$$

since $0 \leq F(x) \leq 1$ (see sketch on board).

The inverse CDF method

$$X = F^{-1}(U).$$

So

$$\begin{aligned}
\Pr(X \le x) &= \Pr\left(F^{-1}(U) \le x\right) \\
&= \Pr\left(F(F^{-1}(U)) \le F(x)\right) \\
&= \Pr(U \le F(x)) \\
&= F(x),
\end{aligned}$$

since $0 \le F(x) \le 1$ (see sketch on board).

The inverse CDF method

Example

Suppose we obtain two observations from a $U(0,1)$ distribution: 0.1 and 0.85. Use these values to obtain two observations from $X \sim Exp(2)$.

Solution

If $X \sim Exp(2)$ then $F_X(x) = 1 - e^{-2x}$.

If $y = 1 - e^{-2x}$, then

$$\begin{aligned} e^{-2x} &= 1 - y \\ -2x &= \log(1 - y) \\ x &= -\frac{1}{2}\log(1 - y). \end{aligned}$$

Plug in $y = 0.1$ and $y = 0.85$; gives realized values of 0.0527 and 0.949.

Solution

If $X \sim Exp(2)$ then $F_X(x) = 1 - e^{-2x}$.

If $y = 1 - e^{-2x}$, then

$$\begin{aligned} e^{-2x} &= 1 - y \\ -2x &= \log(1 - y) \\ x &= -\frac{1}{2}\log(1 - y). \end{aligned}$$

Plug in $y = 0.1$ and $y = 0.85$; gives realized values of 0.0527 and 0.949.

Solution

If $X \sim Exp(2)$ then $F_X(x) = 1 - e^{-2x}$.

If $y = 1 - e^{-2x}$, then

$$\begin{aligned} e^{-2x} &= 1 - y \\ -2x &= \log(1-y) \\ x &= -\frac{1}{2}\log(1-y). \end{aligned}$$

Plug in $y = 0.1$ and $y = 0.85$; gives realized values of 0.0527 and 0.949.

Example

- Patients arriving at a doctor's surgery have appointment times 15 minutes apart.
- The random variable L measures how late a patient is for his/her appointment, with negative values denoting early arrivals.

The PDF of L is

$$f_L(\ell) = \begin{cases} 0.0025(\ell + 20), & -20 \leq \ell \leq 0, \\ 0.05e^{-\ell/10}, & \ell > 0. \end{cases}$$

Questions

a. Sketch $f_L(\ell)$.

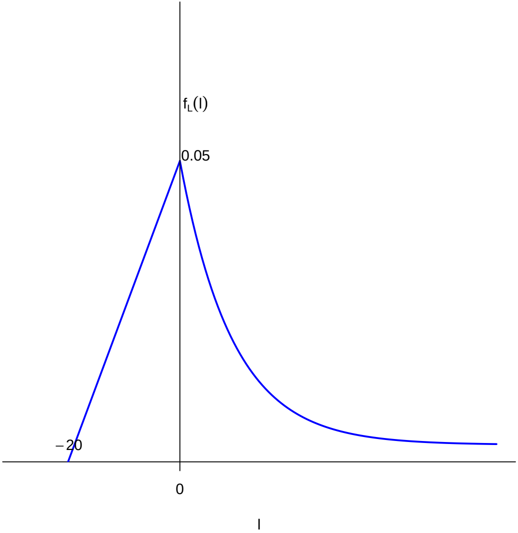
b. Show that the CDF of L is

$$F_L(\ell) = \begin{cases} 0, & \ell < -20, \\ 0.00125(\ell + 20)^2, & -20 \le \ell \le 0, \\ 1 - \frac{e^{-\ell/10}}{2}, & \ell > 0. \end{cases}$$

c. Suppose that 0.205 and 0.713 are two realized values of $U \sim U(0, 1)$. Show that the corresponding realized values of L are -7.19 and 5.55 under the inverse CDF simulation method.

Solution

First, the PDF:



Solution

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \leq \ell \leq 0$:

$$
\begin{aligned}
F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\
&= \int_{-20}^{\ell} 0.0025(s+20)ds \\
&= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\
&= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0.
\end{aligned}
$$

Also, $F_L(0) = 0.5$.

Solution

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \leq \ell \leq 0$:

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0. \end{aligned}$$

Also, $F_L(0) = 0.5$.

Solution

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \leq \ell \leq 0$:

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0. \end{aligned}$$

Also, $F_L(0) = 0.5$.

Solution

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \leq \ell \leq 0$:

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0. \end{aligned}$$

Also, $F_L(0) = 0.5$.

Solution

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \leq \ell \leq 0$:

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0. \end{aligned}$$

Also, $F_L(0) = 0.5$.

Solution

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \le \ell \le 0$:

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \le \ell \le 0. \end{aligned}$$

Also, $F_L(0) = 0.5$.

Solution

Now to find the CDF: L lies between -20 and $+\infty$, so $F_L(\ell) = 0$ when $\ell < -20$.

For $-20 \leq \ell \leq 0$:

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{\ell} f_L(s)ds \\ &= \int_{-20}^{\ell} 0.0025(s+20)ds \\ &= \left[\frac{0.0025}{2}(s+20)^2\right]_{-20}^{\ell} \\ &= 0.00125(\ell+20)^2 \quad \text{for } -20 \leq \ell \leq 0. \end{aligned}$$

Also, $F_L(0) = 0.5$.

Solution

Now

$$
\begin{aligned}
F_L(\ell) &= \int_{-20}^{0} f_L(s)ds + \int_0^{\ell} f_L(s)ds \quad \text{for } \ell > 0 \\
&= 0.5 + \int_0^{\ell} 0.05e^{-s/10} ds \\
&= 0.5 + \left[0.05 \times -10e^{-s/10}\right]_0^{\ell} \\
&= 0.5 + 0.5 - 0.5e^{-\ell/10} \\
&= 1 - \frac{e^{-\ell/10}}{2}, \quad \text{as required.}
\end{aligned}
$$

Solution

Now

$$\begin{aligned}
F_L(\ell) &= \int_{-20}^{0} f_L(s)ds + \int_{0}^{\ell} f_L(s)ds \quad \text{for } \ell > 0 \\
&= 0.5 + \int_0^{\ell} 0.05e^{-s/10}ds \\
&= 0.5 + \left[0.05 \times -10e^{-s/10}\right]_0^{\ell} \\
&= 0.5 + 0.5 - 0.5e^{-\ell/10} \\
&= 1 - \frac{e^{-\ell/10}}{2}, \quad \text{as required.}
\end{aligned}$$

Solution

Now

$$
\begin{aligned}
F_L(\ell) &= \int_{-20}^{0} f_L(s)ds + \int_0^{\ell} f_L(s)ds \quad \text{for } \ell > 0 \\
&= 0.5 + \int_0^{\ell} 0.05e^{-s/10}ds \\
&= 0.5 + \left[0.05 \times -10e^{-s/10}\right]_0^{\ell} \\
&= 0.5 + 0.5 - 0.5e^{-\ell/10} \\
&= 1 - \frac{e^{-\ell/10}}{2}, \quad \text{as required.}
\end{aligned}
$$

Solution

Now

$$\begin{aligned} F_L(\ell) &= \int_{-20}^{0} f_L(s)ds + \int_{0}^{\ell} f_L(s)ds \quad \text{for } \ell > 0 \\ &= 0.5 + \int_{0}^{\ell} 0.05e^{-s/10}ds \\ &= 0.5 + \left[0.05 \times -10e^{-s/10}\right]_0^{\ell} \\ &= 0.5 + 0.5 - 0.5e^{-\ell/10} \\ &= 1 - \frac{e^{-\ell/10}}{2}, \quad \text{as required.} \end{aligned}$$

Solution

Now

$$
\begin{aligned}
F_L(\ell) &= \int_{-20}^{0} f_L(s)ds + \int_{0}^{\ell} f_L(s)ds \quad \text{for } \ell > 0 \\
&= 0.5 + \int_{0}^{\ell} 0.05e^{-s/10} ds \\
&= 0.5 + \left[0.05 \times -10e^{-s/10}\right]_0^{\ell} \\
&= 0.5 + 0.5 - 0.5e^{-\ell/10} \\
&= 1 - \frac{e^{-\ell/10}}{2}, \quad \text{as required.}
\end{aligned}
$$

Solution

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$\begin{aligned} y &= 0.00125(\ell + 20)^2, \quad \text{so} \\ \ell &= \pm\sqrt{\frac{y}{0.00125}} - 20, \end{aligned}$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

Solution

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$\begin{aligned} y &= 0.00125(\ell + 20)^2, \quad \text{so} \\ \ell &= \pm\sqrt{\frac{y}{0.00125}} - 20, \end{aligned}$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

Solution

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$\begin{aligned} y &= 0.00125(\ell + 20)^2, \quad \text{so} \\ \ell &= \pm\sqrt{\frac{y}{0.00125}} - 20, \end{aligned}$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

Solution

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$\begin{aligned} y &= 0.00125(\ell + 20)^2, \quad \text{so} \\ \ell &= \pm\sqrt{\frac{y}{0.00125}} - 20, \end{aligned}$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

Solution

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$y = 0.00125(\ell + 20)^2, \quad \text{so}$$
$$\ell = \pm\sqrt{\frac{y}{0.00125}} - 20,$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

Solution

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$\begin{aligned} y &= 0.00125(\ell + 20)^2, \quad \text{so} \\ \ell &= \pm\sqrt{\frac{y}{0.00125}} - 20, \end{aligned}$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

Solution

Now for some realized values:

$u = 0.205 < 0.5$, so realized value ℓ lies between -20 and 0. Thus

$$F_L(\ell) = 0.00125(\ell + 20)^2.$$

Find the inverse, and evaluate at $u = 0.205$:

$$\begin{aligned} y &= 0.00125(\ell + 20)^2, \quad \text{so} \\ \ell &= \pm\sqrt{\frac{y}{0.00125}} - 20, \end{aligned}$$

giving

$$\ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2.$$

Solution

Similarly, $u = 0.713 > 0.5$ so realized value > 0. Thus

$$F_L(\ell) = 1 - \frac{e^{-\ell/10}}{2}.$$

Find the inverse, and evaluate at $u = 0.713$:

$$\begin{aligned} y &= 1 - \frac{e^{-\ell/10}}{2} \quad \text{so} \\ \ell &= -10\log(2 - 2y), \end{aligned}$$

giving

$$\ell = -10\log(2 - 2 \times 0.713) = 5.55.$$

Solution

Similarly, $u = 0.713 > 0.5$ so realized value > 0. Thus

$$F_L(\ell) = 1 - \frac{e^{-\ell/10}}{2}.$$

Find the inverse, and evaluate at $u = 0.713$:

$$\begin{aligned} y &= 1 - \frac{e^{-\ell/10}}{2} \quad \text{so} \\ \ell &= -10 \log(2 - 2y), \end{aligned}$$

giving

$$\ell = -10 \log(2 - 2 \times 0.713) = 5.55.$$

Solution

Similarly, $u = 0.713 > 0.5$ so realized value > 0. Thus

$$F_L(\ell) = 1 - \frac{e^{-\ell/10}}{2}.$$

Find the inverse, and evaluate at $u = 0.713$:

$$\begin{aligned} y &= 1 - \frac{e^{-\ell/10}}{2} \quad \text{so} \\ \ell &= -10\log(2 - 2y), \end{aligned}$$

giving

$$\ell = -10\log(2 - 2 \times 0.713) = 5.55.$$

Solution

Similarly, $u = 0.713 > 0.5$ so realized value > 0. Thus

$$F_L(\ell) = 1 - \frac{e^{-\ell/10}}{2}.$$

Find the inverse, and evaluate at $u = 0.713$:

$$\begin{aligned} y &= 1 - \frac{e^{-\ell/10}}{2} \quad \text{so} \\ \ell &= -10\log(2-2y), \end{aligned}$$

giving

$$\ell = -10\log(2 - 2 \times 0.713) = 5.55.$$

Solution

Similarly, $u = 0.713 > 0.5$ so realized value > 0. Thus

$$F_L(\ell) = 1 - \frac{e^{-\ell/10}}{2}.$$

Find the inverse, and evaluate at $u = 0.713$:

$$\begin{aligned} y &= 1 - \frac{e^{-\ell/10}}{2} \quad \text{so} \\ \ell &= -10 \log(2 - 2y), \end{aligned}$$

giving

$$\ell = -10 \log(2 - 2 \times 0.713) = 5.55.$$

Recap Quiz

1. Write down R code that simulates 100 observations from the $U(0,1)$ distribution and stores the output in the vector `U`.
2. Suppose X has CDF

$$F_X(x) = 1 - \frac{1}{x}, \quad x \geq 1.$$

   Find $F_X^{-1}(x)$.

3. Write an R function called `inv.cdf` to return the inverse of $F_X(x)$.
4. Write down a single line of R code to generate 100 realizations of the random variable X.
5. Write down a single line of R code to produce a histogram of your realizations of X, coloured yellow and with a suitable title.