Lecture 2: Inverse CDF method
In this lecture we look at the **inverse CDF method** for simulating from continuous random variables.
The inverse CDF method

This is a method for simulating univariate continuous random variables

- Let $U \sim U(0, 1)$
- Suppose $F(x)$ is a well-defined CDF which is invertible
- Then the random variable $X = F^{-1}(U)$ has CDF $F(x)$
The inverse CDF method

\[ X = F^{-1}(U). \]

So

\[ \Pr(X \leq x) = \Pr(F^{-1}(U) \leq x) \]

\[ = \Pr(F(F^{-1}(U)) \leq F(x)) \]

\[ = \Pr(U \leq F(x)) \]

\[ = F(x), \]

since \( 0 \leq F(x) \leq 1 \) (see sketch on board).
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The inverse CDF method

Example

Suppose we obtain two observations from a $U(0, 1)$ distribution: 0.1 and 0.85. Use these values to obtain two observations from $X \sim \text{Exp}(2)$. 
If $X \sim \text{Exp}(2)$ then $F_X(x) = 1 - e^{-2x}$.

If $y = 1 - e^{-2x}$, then

\[ e^{-2x} = 1 - y \]

\[ -2x = \log(1 - y) \]

\[ x = -\frac{1}{2} \log(1 - y). \]

Plug in $y = 0.1$ and $y = 0.85$; gives realized values of 0.0527 and 0.949.
If $X \sim \text{Exp}(2)$ then $F_X(x) = 1 - e^{-2x}$.

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Example

- Patients arriving at a doctor’s surgery have appointment times 15 minutes apart.
- The random variable $L$ measures how late a patient is for his/her appointment, with negative values denoting early arrivals.

The PDF of $L$ is

$$f_L(\ell) = \begin{cases} 
0.0025(\ell + 20), & -20 \leq \ell \leq 0, \\
0.05e^{-\ell/10}, & \ell > 0.
\end{cases}$$
Questions

a. Sketch $f_L(\ell)$.

b. Show that the CDF of $L$ is

$$F_L(\ell) = \begin{cases} 
0, & \ell < -20, \\
0.00125(\ell + 20)^2, & -20 \leq \ell \leq 0, \\
1 - \frac{e^{-\ell/10}}{2}, & \ell > 0.
\end{cases}$$

c. Suppose that 0.205 and 0.713 are two realized values of $U \sim U(0, 1)$. Show that the corresponding realized values of $L$ are $-7.19$ and $5.55$ under the inverse CDF simulation method.
First, the PDF:
Now to find the CDF: \( L \) lies between \(-20\) and \(+\infty\), so \( F_L(\ell) = 0 \) when \( \ell < -20 \).

For \(-20 \leq \ell \leq 0\):

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F_L(\ell) = \int_{-20}^{\ell} f_L(s)ds
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= \int_{-20}^{\ell} 0.0025(s + 20)ds
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= 0.00125(\ell + 20)^2 \quad \text{for} \ -20 \leq \ell \leq 0.
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Also, \( F_L(0) = 0.5 \).
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Now

\[ F_L(\ell) = \int_{-20}^{0} f_L(s) \, ds + \int_{0}^{\ell} f_L(s) \, ds \quad \text{for } \ell > 0 \]

\[ = 0.5 + \int_{0}^{\ell} 0.05e^{-s/10} \, ds \]

\[ = 0.5 + \left[ 0.05 \times -10e^{-s/10} \right]_{0}^{\ell} \]

\[ = 0.5 + 0.5 - 0.5e^{-\ell/10} \]

\[ = 1 - \frac{e^{-\ell/10}}{2}, \quad \text{as required.} \]
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Solution

Now for some realized values:

\[ u = 0.205 < 0.5, \text{ so realized value } \ell \text{ lies between } -20 \text{ and } 0. \text{ Thus } \]

\[ F_L(\ell) = 0.00125(\ell + 20)^2. \]

Find the inverse, and evaluate at \( u = 0.205: \)

\[ y = 0.00125(\ell + 20)^2, \text{ so } \]

\[ \ell = \pm \sqrt{\frac{y}{0.00125}} - 20, \]

giving

\[ \ell = \sqrt{\frac{0.205}{0.00125}} - 20 = -7.2. \]
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Similarly, \( u = 0.713 > 0.5 \) so realized value \( > 0 \). Thus

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Find the inverse, and evaluate at \( u = 0.713 \):

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y = 1 - \frac{e^{-\ell/10}}{2} \quad \text{so} \quad \ell = -10 \log(2 - 2y),
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\ell = -10 \log(2 - 2 \times 0.713) = 5.55.
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1. Write down R code that simulates 100 observations from the $U(0, 1)$ distribution and stores the output in the vector $U$.

2. Suppose $X$ has CDF

$$F_X(x) = 1 - \frac{1}{x}, \quad x \geq 1.$$ 

Find $F_X^{-1}(x)$.

3. Write an R function called `inv.cdf` to return the inverse of $F_X(x)$.

4. Write down a single line of R code to generate 100 realizations of the random variable $X$.

5. Write down a single line of R code to produce a histogram of your realizations of $X$, coloured yellow and with a suitable title.