Lecture 3: Transformation of Random Variables

Recap Quiz

1. Write down R code that simulates 100 observations from the $U(0,1)$ distribution and stores the output in the vector `U`.
2. Suppose X has CDF

$$F_X(x) = 1 - \frac{1}{x}, \quad x \geq 1.$$

   Find $F_X^{-1}(x)$.

3. Write an R function called `inv.cdf` to return the inverse of $F_X(x)$.
4. Write down a single line of R code to generate 100 realizations of the random variable X.
5. Write down a single line of R code to produce a histogram of your realizations of X, coloured yellow and with a suitable title.

Transforming a normal random variable

- Consider how we might model a continuous positive random quantity e.g. blood pressure or height
- The Normal distribution is not suitable!

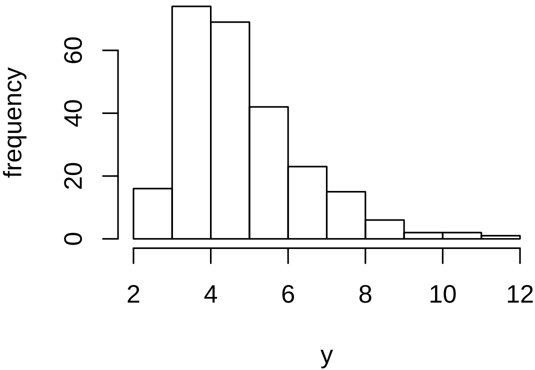
One way to obtain a positive random variable is to define

$$Y = \exp(X), \quad \text{where } X \sim N(\mu, \sigma^2).$$

Then Y is a random variable which is a transformation of X.

Simulation to explore transformation

```r
y = exp(rnorm(250, mean = 1.5, sd = 0.3))
hist(y, xlab = 'Y', ylab = 'frequency')
```



Calculating PDFs

- This looks like a useful distribution – what is its PDF?
- $Y = \exp(X)$ so $X = \log(Y)$
- So $f_Y(y) = f_X(\log(y))$ is a good guess, but is NOT the right answer!

Example

If $X \sim U(0, 1)$, what is the PDF of $Y = X^3$?

Solution

$$
\begin{aligned}
F_Y(y) &= \Pr(Y \le y) \\
&= \Pr\left(X^3 \le y\right) \\
&= \Pr\left(X \le y^{1/3}\right) \\
&= F_X(y^{1/3}).
\end{aligned}
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Therefore

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_X(y^{1/3}) \\
&= f_X(y^{1/3}) \times \frac{1}{3} y^{-2/3} \quad \text{by chain rule} \\
&= \frac{1}{3} y^{-2/3}, \quad 0 \le y \le 1 \text{ (and zero otherwise).}
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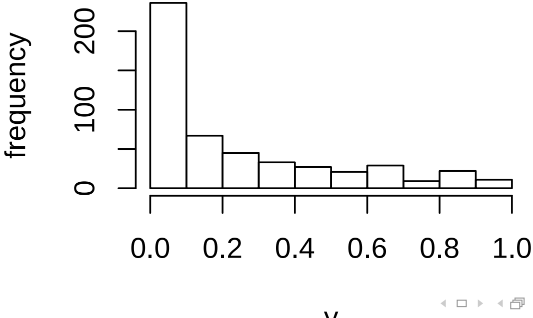
Checking via simulation

To confirm this result, we can simulate Y:

```r
y = (runif(500))^3
hist(y, xlab = 'Y', ylab = 'frequency')
```



Further examples

Example

If $X \sim N(0,1)$, what is the PDF of $Y = X^2$?

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Further examples

Example

If $X \sim N(\mu, \sigma^2)$, what is the PDF of $Y = \exp(X)$?
This is the example used at the start of the lecture. We say Y has the log-normal distribution and write $Y \sim LN(\mu, \sigma^2)$.

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\begin{aligned}
F_Y(y) &= F_X(\log y) \quad \text{by usual argument with CDF} \\
f_Y(y) &= \frac{d}{dy} F_X(\log y) \\
&= f_X(\log y) \times \frac{1}{y} \quad \text{for } y \geq 0 \\
&= \frac{1}{\sigma y \sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left(\frac{(\log y - \mu)^2}{\sigma^2} \right) \right\}.
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The univariate transformation method

If X has pdf $f_X(x)$ and $h(x)$ is a monotonically increasing or monotonically decreasing function then

- h is invertible and
- the PDF of $Y = h(X)$ is

$$f_Y(y) = f_X(h^{-1}(y)) \times \left| \frac{d\left(h^{-1}(y)\right)}{dy} \right|$$

when y is in the range of $h(x)$ and zero otherwise.

This result allows direct calculation of $f_Y(y)$ and is called the univariate transformation method.

See whiteboard for Proof.

Some Examples

Example

When $X \sim U(0,1)$, use the transformation method to find the PDF of $Y = X^3$.

$$h(x) = x^3, \quad h^{-1}(y) = y^{1/3}.$$

$$\begin{aligned} f_Y(y) &= 1 \times \frac{d}{dy} y^{1/3} \\ &= \frac{1}{3} y^{-2/3}, \quad 0 \le y \le 1. \end{aligned}$$

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Some examples

Example

When $X \sim U(-1,1)$ and $Y = \exp(X)$, use the transformation method to find the PDF of Y.

$$h(x) = e^x, \quad h^{-1}(y) = \log(y).$$

$$\begin{aligned} f_Y(y) &= \frac{1}{2} \times \left| \frac{d}{dy} \log(y) \right| \\ &= \frac{1}{2|y|} = \frac{1}{2y}, \quad e^{-1} \leq y \leq e^1, \text{ zero otherwise.} \end{aligned}$$

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Example

Use the transformation method to find the PDF of the log-normal distribution.

$$h(x) = e^x, \quad h^{-1}(y) = \log(y).$$

$$\begin{aligned} f_Y(y) &= f_X(\log(y)) \times \frac{1}{y} \qquad \text{for } y > 0 \\ &= \frac{1}{y} \times \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\log(y)-\mu}{\sigma}\right)^2\right]. \end{aligned}$$

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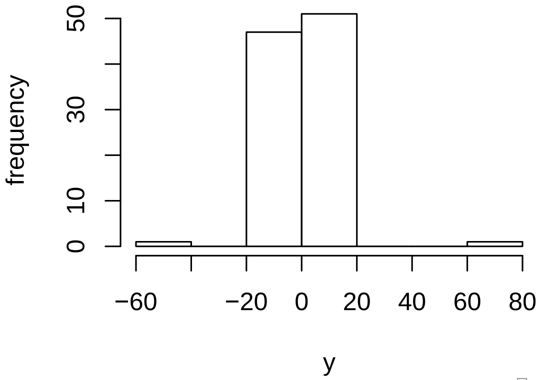
Cauchy example

Example

Suppose $X \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$ and $Y = \tan(X)$.
Use simulation to study the distribution of Y and explicitly compute the PDF.

Solution

```r
y = tan(runif(100, -pi/2, pi/2))
hist(y, xlab = 'y', ylab = 'frequency', main = '')
```



Solution

This distribution looks odd! Calculate the PDF:

$$h(x) = \tan(x), \quad h^{-1}(y) = \tan^{-1}(y).$$

$$\begin{aligned} f_Y(y) &= \frac{1}{\pi}\frac{d}{dy}\left(\tan^{-1} y\right) \quad -\infty < y < \infty, \\ &= \frac{1}{\pi(1+y^2)} \quad -\infty < y < \infty. \end{aligned}$$

This is the **Cauchy distribution**. It has *very* heavy tails.

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