2 Inverse CDF method for simulating random variables

Let $U \sim U(0,1)$. Suppose $F(x)$ is a well-defined CDF which is invertible. Then the random variable $X = F^{-1}(U)$ has CDF $F(x)$.

Proof:

This result enables us to simulate the random variable X very readily provided we can simulate uniform random variables. In R we would make use of the command:

```
x = inv.cdf(runif(1))
```

Here, the function `inv.cdf` is a function that we have written ourselves to return the inverse of the CDF of a particular distribution of interest; see **Example 2.2** for more details.

Example 2.1: Suppose we obtain two observations from a $U(0,1)$ distribution: 0.1 and 0.85. Use these values to obtain two observations from $X \sim Exp(2)$.

Example 2.2: Write some R code to sample n observations from an $Exp(2)$ distribution. Do not use the built-in `rexp` function. Check your code by comparing a histogram of some simulated realisations against the PDF for an Exp(2) distribution. Also check your code against results from the built-in `rexp` function.

Example 2.3: Patients arriving at a doctor's surgery have appointment times 15 minutes apart. The random variable L measures how late a patient is for his/her appointment, with negative values denoting early arrives. The PDF of L is

$$f_L(l) = \begin{cases} 0.0025(l+20), & -20 \leq l \leq 0, \\ 0.05e^{-l/10}, & > 0. \end{cases}$$

(a) Sketch $f_L(l)$.

(b) Show that the CDF of L is

$$F_L(l) = \begin{cases} 0, & l < -20, \\ 0.00125(l+20)^2, & -20 \leq l \leq 0, \\ 1 - \frac{e^{-l/10}}{2}, & l > 0. \end{cases}$$

(c) Suppose that 0.205 and 0.713 are two realized values of $U \sim U(0,1)$. Show that the corresponding realized values of L are -7.19 and 5.55 under the inverse CDF simulation method.

Example 2.4: The Weibull distribution is often used in survival data analysis to model lifetimes (e.g. the lifetimes of patients receiving a particular drug). If $X \sim Weibull(\lambda, \kappa)$, then it has cumulative distribution function

$$F_X(x; \lambda, \kappa) = 1 - e^{-(x/\lambda)^\kappa}, \quad x \geq 0, \lambda, \kappa > 0.$$

Write a function in R to produce realisations from this distribution.