CASE STUDY 2

MAS2317/3317: Introduction to Bayesian Statistics

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BAYESIAN MODELLING OF EXTREME RAINFALL DATA

1. MOTIVATION

Over the last 30 years or so, interest in the use of statistical methods for modelling environmental extremes has grown dramatically, for good reason: climate change has resulted in an increase in severity, and frequency, of environmental phenomena resulting in huge economic loss, and loss of human life. For example, Hurricane Katrina (see Figure 1) hit southern states of the USA in September 2005, killing nearly 2000 people, displacing well over one million people, and costing the US economy an estimated $110 billion. This was the "storm of the century" – that which we could expect to see once in a hundred years – and yet just a few weeks later, Hurricane Rita, a storm of similar ferocity, battered Texas and Louisiana.

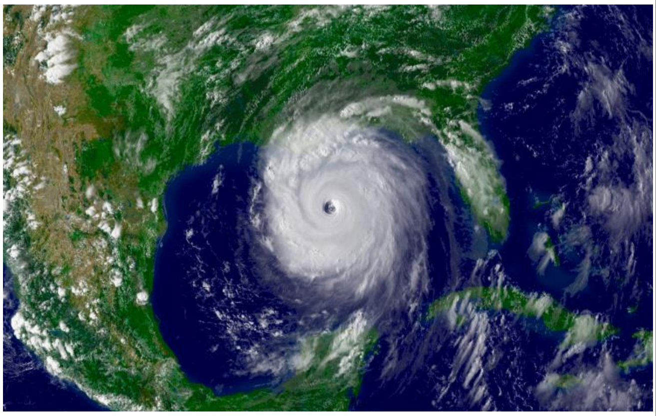

FIGURE 1: Satellite image of Hurricane Katrina

In other parts of the world, the frequency and severity of periods of extreme drought have caused widespread famine, resulting in massive loss of life in parts of Sub-Saharan Africa; the frequency and severity of severe cold spells in parts of Eastern Russia and China have made it difficult to stockpile adequate fuel supplies for the winter period; and rapid shifts in climate on a micro-scale have resulted in an increase in mass land movements such as landslides and avalanches (e.g. landslides in Caracas, Venezuela (2010) and avalanches in Switzerland since 2001).

Closer to home, the U.K. has had its fair share of storms over the last quarter of a century. The "great storm" of October 1987 battered parts of Southern England with hurricane-strength wind speeds, causing 22 deaths and £7.3 billion worth of damage. Although weather forecasting systems in place at the time *did* warn of a spell of unsettled weather, the U.K. Met Office did not foresee the extreme wind speeds and rainfall that this storm brought (see the BBC's Michael Fish weather forecast video on the slides for this case study). This was also dubbed the U.K.'s "storm of the century", and yet just two years later similar strength wind speeds were observed during another storm across central and southern parts of the U.K., again resulting in a number of deaths and huge financial loss.

More recently, the U.K. has also seen a dramatic rise in extreme flooding events. The pictures shown in Figure 2 below are from floods in central and south-western England in 2007 – 2009; not only do such events pose significant risk to life, but the rise in frequency and severity of such flooding events in the U.K. has made the price of flooding insurance premiums skyrocket.

FIGURE 2: Various images of flooding in U.K. towns 2007 - 2009

2. STATISTICAL MODELLING OF EXTREME RAINFALL DATA

An increase in the types of environmental events discussed in Section 1 has, over the last 30 years, seen an increase in the use of statistical models for estimating the strength, and associated frequency, of storms, floods, droughts, etc. We will now focus on flooding events in the U.K., and so our interest lies in the occurrence of extreme rainfall data. In particular, in 2003 the U.K. Met Office supplied us with daily rainfall totals (in mm) for a network of 204 sites across the U.K., collected between 1961 and 1995 (inclusive). For each site, this gives nearly 13,000 rainfall observations. However, many of these daily totals are zero values (days on which there was no rainfall recorded at all) – data which we have no interest in, since it is extreme rainfall totals that cause the flooding discussed in Section 1. Further, we are only interested in the extreme rainfall totals – the rest of the data, in terms of what causes flooding, is of no interest.

One way forward here is to extract the largest daily rainfall total from each year, resulting in a set of 35 *annual maxima* for each of our 204 sites. By doing this, every single observation in our extracted set is now "extreme", and we can consider using specialist statistical models for "extreme values". Figure 3 below shows the set of daily rainfall totals obtained for one site – Oxford – with the extracted set of annual maxima shown underneath.

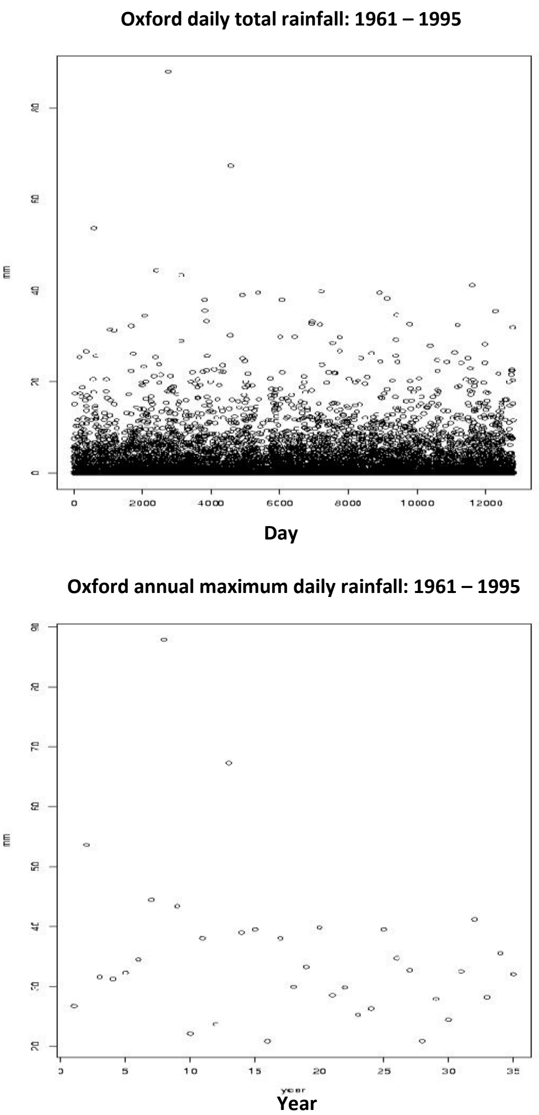

FIGURE 3: Daily rainfall totals, and extracted annual maxima, for Oxford

2.1 A statistical model for extremes

The **Generalised Extreme Value distribution** (GEV) was independently derived by R. von Mises (1954) and A.F. Jenkinson (1955). This is a limiting model for extremes of a stationary series, with cumulative distribution function given by

$$F_X(x|\mu,\sigma,\gamma) = \exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\gamma}\right\}, \qquad \textbf{(1)}$$

Where μ, σ and γ are the location, scale and shape parameters of the distribution. In practice, the extremes X are often obtained as the set of annual maxima, as extracted for the Oxford rainfall dataset in Figure 3. The most commonly used method for fitting Equation (1) to our set of annual maxima is maximum likelihood; although there are no closed-form solutions for the MLEs $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\gamma}$, a Newton-Raphson type procedure can be used in **R** to maximise the likelihood; for the Oxford rainfall extremes shown in Figure 3 (bottom), this gives $\hat{\mu}$ = 40.8 (1.58), $\hat{\sigma}$ = 9.7 (1.19) and $\hat{\gamma}$ = 0.1 (0.11), with standard errors given in brackets. Figure 4 shows a histogram of the Oxford rainfall extremes with the fitted GEV superimposed.

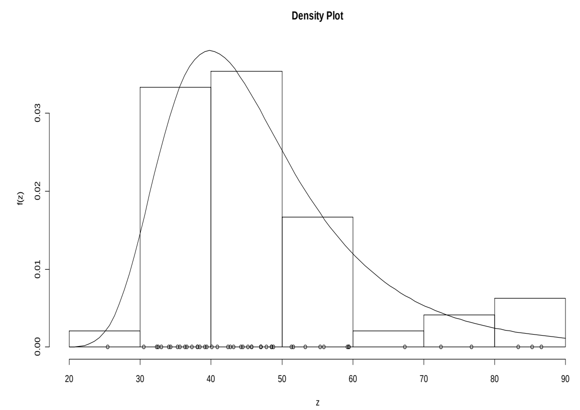

FIGURE 4: Histogram of the Oxford rainfall maxima with fitted GEV superimposed

2.2 Practical use of the GEV

So we have a statistical model for extremes which seems to fit our annual maximum daily rainfall data quite well. So what? One practical application of such a model is to aid the design of flood defences. For example, suppose we wish to protect a town – perhaps Oxford – against a flooding event we would expect to occur once every hundred years. We only have 35 years worth of data, so – in effect – we're trying to estimate a flooding event which is *more extreme than has ever occurred before*. This requires *extrapolation beyond the range of our data*, and since there is both a theoretical and practical basis for using the GEV, we can estimate such quantities by calculating high quantiles from our fitted distribution.

For example, for the Oxford dataset, we would solve the following equation for $\hat{z}_{100}$:

$$\exp\left\{-\left[1+0.1\left(\frac{\hat{z}_{100}-40.8}{9.7}\right)\right]^{-1/0.1}\right\} = 0.99, \qquad \textbf{(2)}$$

where $\hat{z}_{100}$ is known as the **100 year return level**. Thus, a flood defence system – such as that shown in Bewdley in

Figure 5 – would need to be tall enough to withstand a daily rainfall total of at least $\hat{z}_{100}$ mm. Obviously, the calculation of the height of the flood defence would have to take into account the accumulation of successive daily rainfall totals $\hat{z}_{100}$ mm, and so the height of the defence would be a function of $\hat{z}_{100}$ and the actual duration of a storm event.

FIGURE 5: Flood barriers along the River Severn in Bewdley

Generically, solving equation (2) for $\hat{z}_r$, the ***r* – year return level**, gives

$$\hat{z}_r = \hat{\mu} + \frac{\hat{\sigma}}{\hat{\gamma}}\left\{\left[-\ln\left(1-\frac{1}{r}\right)\right]^{-\hat{\gamma}} - 1\right\}. \qquad \textbf{(3)}$$

Table 1 below shows the estimated 50, 100 and 1000-year return levels for Oxford, along with estimated standard errors; 95% confidence intervals can be found using

$$\hat{z}_r \pm 1.96 \times \text{s.e.}(\hat{z}_r).$$

So, for example, the 95% confidence interval for the 100 year return level at Oxford is (66.20, 109.64)mm; the 1000 year return level has the 95% confidence intervals (58.35, 222.33)mm.

***r* (years)**	**10**	**50**	**200**	**1000**
$\hat{z}_r$ (mm)	65.54	87.92	98.64	140.34
St. error	4.53	11.48	16.22	41.83

TABLE 1: Estimated return levels for Oxford

3. A BAYESIAN PERSPECTIVE

3.1 Advantages of a Bayesian approach

One drawback with the approach outlined in Section 2.1 – and demonstrated in Section 2.2 – is that, once we have extracted our extremes, we are left with just 35 observations (from a dataset of nearly 13,000!). One manifestation of working with such a small set of extremes is that our uncertainty in estimates of parameters – and hence return levels - is rather large. For example, we expect the true 1000 year return level to lie anywhere between about 58mm and 222mm (with 95% confidence). What are we to tell the engineers who are designing a new flood defence system? "Design your flood defence so that it will withstand a daily rainfall maximum of somewhere between 58mm and 222mm"? In my experience, engineers do not like the idea of uncertainty at all, never mind such large uncertainty!

One feature of a Bayesian analysis is that it allows us to incorporate additional information – preferably that provided by an expert – into our analysis. Doing so often informs our analysis greatly, combining prior belief about patterns in extreme rainfall, or wind speeds (for example) with observed data on such processes. This often leads to greater precision in estimates of model parameters and – crucially – estimates of return levels; worth its weight in gold when we are working with such small datasets! However, as we shall see, formulating the beliefs of an expert into prior distributions for the parameters in the GEV can be difficult.

3.2 Obtaining prior distributions for the GEV parameters

3.2.1 Bring in the Expert!

Duncan Reede is an independent consulting hydrologist with over 30 years experience. He graduated with a PhD in Applied Science from Newcastle University in 1977. No matter how much experience an expert might have, it might be too much to ask them to express their beliefs about our model parameters in terms of probability distributions. For example, it *might* be plausible for us to convert Dr. Reede's beliefs about the average maximum daily rainfall accumulation at a site (e.g. Oxford), and from this specify a probability distribution for μ. However, we also need priors for σ and γ, the scale and shape parameters (respectively). How can we get our expert to coherently specify prior beliefs concerning the "shape" of the annual maximum daily rainfall accumulation?

One way around this difficulty is to re-express our distribution in terms of parameters that the expert might feel comfortable with – perhaps return levels! We asked Dr. Reede about the daily rainfall accumulation he might expect to see at Oxford once in ten years (i.e. the 10-year return level). He tells us:

"I have substantial knowledge of rainfall patterns in this region of the UK. For the storm we can expect to see once in ten years – quite a severe storm – I think we could expect to see a daily rainfall accumulation of about 60-65mm, although I'm prepared to go down to 50mm and up to, perhaps, 80mm"

Using the Trial Roulette Method in the *MATCH Uncertainty Elicitation Tool*, we then asked Dr. Reede to distribute chips in bins between 50mm and 80mm to illustrate his uncertainty about z_{10}. This results in

$z_{10} \sim Ga(126, 2)$.

Similarly, for two other return levels, we get:

$z_{50} \sim Ga(242, 2.5)$ and

$z_{200} \sim Ga(180, 1.5)$.

These prior distributions are shown in Figure 6.

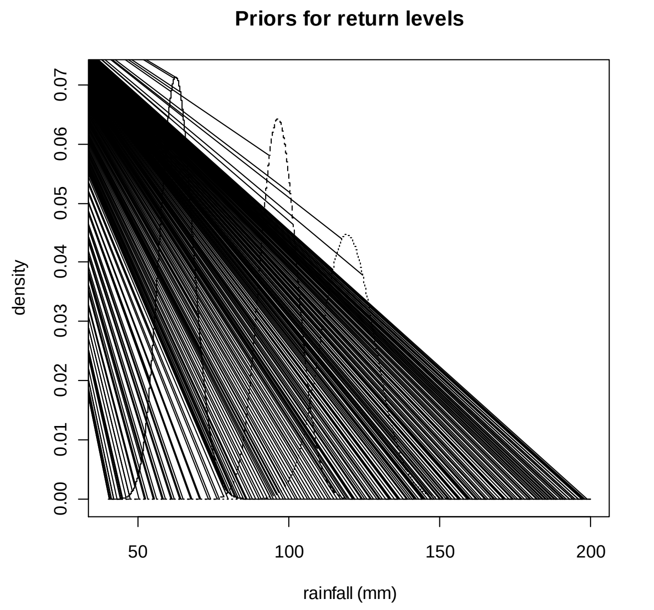

FIGURE 6: Plots of elicited prior distributions for the 10 (solid), 50 (dashed) and 200 (dotted) return levels

<u>3.2.2 Converting to priors for the GEV parameters</u>

We can use the result from Distribution Theory, given in Equation 3.7 of the lecture notes, to "convert" our prior distributions for z_{10}, z_{50} and z_{200} into a prior distribution for (μ, σ, γ):

$$\pi(\mu,\sigma,\gamma) \propto \pi_{z_{10}}(z_{10})\pi_{z_{50}}(z_{50})\pi_{z_{200}}(z_{200}) \times J(\mu,\sigma,\gamma), \quad \textbf{(4)}$$

where $J(\mu, \sigma, \gamma)$ is the *Jacobian determinant*, as discussed in the Thoerem in Section 3.5 of the lecture notes.

Applying Equation (4) to the priors elicited from Dr. Reede in Section 3.2.1 gives a (joint) prior for (μ, σ, γ) which is improper and non-conjugate for the GEV.

3.3 Obtaining the posterior distributions

Since we have non-conjugate prior for the GEV, we cannot proceed to find the posteriors in the same way as we do in the lecture notes for MAS2317. In fact, we used a procedure called Markov Chain Monte Carlo (MCMC) in order to obtain the posteriors for each of μ, σ and γ (see *MAS3321: Bayesian Inference*). This gives samples from the posteriors of μ, σ and γ, we then apply Equation (3) to our samples of these parameters to obtain posterior distributions for return levels of interest.

3.4 Results showing the effect of using the expert priors

Combining the opinion of our expert hydrologist – Dr. Duncan Reede – with the data on extreme rainfall at Oxford (see Section 2) gives the posterior means for the 10, 50, 200 and 1000 year return levels shown in Table 2. Also shown are the posterior standard deviations.

***r* (years)**	**10**	**50**	**200**	**1000**
$E(\hat{z}_r \vert x)$	64.21	91.05	110.31	150.73
$SD(\hat{z}_r \vert x)$	2.14	6.31	8.05	14.79

TABLE 2: Posterior summaries for some return levels at Oxford (units in mm)

Notice that, when you compare the posterior means in Table 2 to the frequentist estimates from table 1, the means have been shifted to take into account the prior beliefs of Dr. Reede as shown in Figure 6. Also notice the dramatic reduction in posterior standard deviation after taking into account the expert's beliefs. Recall that, after extracting our extremes, we were left with just 35 observations – and so our standard errors were fairly large. Augmenting our analysis to include the beliefs of an expert hydrologist has allowed us to reduce our uncertain about our estimates of return levels – crucial information for marine engineers! Table 3 below compares 95% frequentist confidence intervals to their Bayesian counterparts – the reduced width of the intervals in the Bayesian analysis reflects our increased certainty about our estimates.

***r* (years)**	**10**	**50**	**200**	**1000**
Frequentist	(56.7,74.4)	(65.4,110.4)	(66.8,130.5)	(58.3,222.4)
Bayesian	(60.0,68.4)	(78.7,103.4)	(94.5,126.1)	(121.7,179.7)

TABLE 3: 95% Frequentist confidence intervals for some return levels at Oxford versus their Bayesian counterparts

4. CONCLUSIONS

The main aim of this case study was to highlight the advantages of using a Bayesian approach to inference in one particular area of statistics – the study of extreme values. We have seen that, by incorporating the beliefs of an expert hydrologist, we can greatly increase our precision of estimates of return levels – a key parameter used in the design of flood defence systems. However, this is not without difficulties.

It can be extremely difficult to convert an expert's opinion about a particular phenomenon (in this case, extreme rainfall) into something meaningful about a shape parameter or a scale parameter. One way around this is to re-parameterise the model and ask the expert to provide us with information about something he/she feels much more comfortable and "natural" with – in this case, flood levels the hydrologist would expect to see once in 10, 50 and 200 years. We made use of the *MATCH* software to help with this.

We can transform these prior distributions into prior distributions for the model parameters themselves, and then proceed to perform a Bayesian inference for extreme rainfall.

The statistician would then usually provide marine engineers with their results – this information would be used to help design flood defence systems to better protect communities against flood events in the future.