Lecture 6

CONDITIONAL PROBABILITY

Prize winner!

Last week's **prize question winner** is

Alexandra Caskie

Alexandra can choose from:

- Wine
- Beer
- Book voucher

In this chapter, we look at more complicated notions of probability, and extend the multiplication rule for probability to cater for events that are **not independent**.

So far we have only considered probabilities of single events or of several independent events, like two rolls of a die.

For example, on two rolls of a fair, six–sided die, the probability that I roll two sixes is

$$\begin{aligned} P(\text{two sixes}) &= P(\text{six and six}) \\ &= P(\text{six}) \times P(\text{six}) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36}. \end{aligned}$$

A washing basket contains two **green** socks, three **yellow** socks, an **orange** sock and a **purple** sock.

Two socks are drawn at random, without replacement. Obtain the following probabilities:

$$P(\text{two } \textbf{green} \text{ socks}) =$$

$$P(\textbf{orange} \text{ and } \textbf{yellow}) =$$

A washing basket contains two **green** socks, three **yellow** socks, an **orange** sock and a **purple** sock.

Two socks are drawn at random, without replacement. Obtain the following probabilities:

$$P(\text{two } \textbf{green} \text{ socks}) = \frac{2}{7} \times \frac{1}{6} = \frac{1}{21}$$

$$P(\textbf{orange} \text{ and } \textbf{yellow}) = \left(\frac{1}{7} \times \frac{3}{6}\right) + \left(\frac{3}{7} \times \frac{1}{6}\right) = \frac{1}{7}$$

However, in reality, many events are **related**. For example, the probability of it raining in 5 minutes time is dependent on whether or not it is raining now.

We need a mathematical notation to capture how the probability of one event depends on other events taking place.

We do this as follows:

Consider two events A and B. We write

$$P(A|B)$$

for the probability of A given that B has **already happened**.

We describe $P(A|B)$ as the **conditional probability** of A given B.

For example, the probability of it raining in 5 minutes time given that it is raining now would be

$$P(\text{Rain in 5 minutes}|\text{Raining now}).$$

Utility companies need to be able to forecast periods of high demand. They describe their forecasts in terms of probabilities.

Gas and electricity suppliers might relate them to air temperature.

For example,

$$P(\text{High demand}|\text{air temperature is below normal}) = 0.6$$
$$P(\text{High demand}|\text{air temperature is normal}) = 0.2$$
$$P(\text{High demand}|\text{air temperature is above normal}) = 0.05.$$

We can calculate these conditional probabilities using the formula

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)},$$

that is, in terms of the probability of **both events occurring**, $P(A \text{ and } B)$, and the probability of the event that has **already taken place**, $P(B)$.

To see how this formula works, let's consider a simple example based on the class of students in Exercises 5.

Sex	Height (m)	Weight (kg)	Shoe Size	Sex	Height (m)	Weight (kg)	Shoe Size
M	1.91	70	11.0	M	1.78	76	8.5
F	1.73	89	6.5	M	1.88	64	9.0
M	1.73	73	7.0	M	1.88	83	9.0
M	1.63	54	8.0	M	1.70	55	8.0
F	1.73	58	6.5	M	1.76	57	8.0
M	1.70	60	8.0	M	1.78	60	8.0
M	1.82	76	10.0	F	1.52	45	3.5
M	1.67	54	7.5	M	1.80	67	7.5
F	1.55	47	4.0	M	1.92	83	12.0

Suppose we want the probability that a student chosen at random from this class will be **female given that** the student's **shoe size is less than 8**.

$$P(\text{Female}|\text{Shoe size less than 8}) = \frac{4}{7}.$$

This probability can also be calculated using the above formula as follows:

$$\begin{aligned} P(\text{Shoe size} < 8) &= \frac{7}{18} \\ P(\text{Shoe size} < 8 \text{ and female}) &= \frac{4}{18} \end{aligned}$$

and so

$$\begin{aligned} P(\text{Female}|\text{Shoe size} < 8) &= \frac{P(\text{Shoe size} < 8 \text{ and female})}{P(\text{Shoe size} < 8)} \\ &= \frac{4/18}{7/18} \\ &= \frac{4}{7}. \end{aligned}$$

Multiplication of probabilities

We saw in Chapter 5 that, if two events A and B are **independent**, then

$$P(A \text{ and} B) \quad = \quad P(A) \times P(B).$$

Now we know that

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)},$$

we can easily see that

$$P(A \text{ and } B) = P(B) \times P(A|B).$$

Of course it is also true that $P(A \text{ and } B) = P(A) \times P(B|A)$.

For example, consider a student chosen at random from the example class.

- Let F be the event "**the student is female**"
- Let S be the event "**the student's weight is less than 60kg**"

Then the probability that the student is female **and** has a weight less than 60kg is

$$\begin{aligned} P(F \text{ and } S) &= P(S) \times P(F|S) = \frac{7}{18} \times \frac{3}{7} = \frac{3}{18} \\ &= P(F) \times P(S|F) = \frac{4}{18} \times \frac{3}{4} = \frac{3}{18}. \end{aligned}$$

Notice that, if M is the event "**the student is male**", then

$$\begin{aligned} P(S|M) &= 4/14 \\ &= 0.286 \end{aligned}$$

and this is **not equal to** $P(S|F) = 3/4 = 0.75$.

So the probability of the student having a weight less than 60kg depends on the student's sex. The events S and F are **not independent**.

Similarly, $P(F|S) = 3/7 = 0.429$ while $P(F|L) = 1/11 = 0.091$, where L is the event "**the student's weight is not less than 60kg**".

Let $\bar{B}$ be the event "**not** B".

So, for example, $\bar{F} = M$.

Then we say that two events A and B are independent if

$$P(A|B) \;=\; P(A|\bar{B}) = P(A).$$

It is easy to show that this is equivalent to

$$P(B|A) \;=\; P(B|\bar{A}) = P(B).$$

If A and B are independent, then

$$P(A \text{ and } B) \;=\; P(A) \times P(B).$$

For example, consider the following probabilities for customers at a cafe who can choose either ice cream or treacle sponge and custard:

	Ice cream	Treacle sponge
Male	0.250	0.150
Female	0.375	0.225

We see that

$$\begin{aligned} P(\text{male}) &= 0.250 + 0.150 \\ &= 0.4, \end{aligned}$$

and

$$\begin{aligned} P(\text{female}) &= 0.375 + 0.225 \\ &= 0.6 \\ &= 1 - P(\text{male}). \end{aligned}$$

Now

$$P(\text{Ice cream}|\text{Male}) = \frac{0.250}{0.4} = 0.625$$

and

$$P(\text{Ice cream}|\text{Female}) = \frac{0.375}{0.6} = 0.625,$$

so Ice cream and Male are independent events.

In fact, the variables **Sex** and **Dessert–choice** are independent in this example.

So the probability that a customer is male and chooses ice cream is just

$$\begin{aligned} P(\text{Male}) \times P(\text{Ice cream}) &= 0.4 \times 0.625 \\ &= 0.25 \end{aligned}$$

Tree Diagrams

Tree diagrams or **probability trees** are simple ways of presenting probabilistic information.

Let us first consider a simple example in which a die is rolled twice.

Suppose we are interested in the probability that we score a six on both rolls.

This probability can be calculated as

$$\begin{aligned} P(6 \text{ and } 6) &= P(6 \text{ on 1st}) \times P(6 \text{ on 2nd} | 6 \text{ on 1st}) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36}. \end{aligned}$$

This example can be represented as a **tree diagram** in which experiments are represented by circles (called **nodes**) and the outcomes of the experiments as **branches**.

The branches are annotated by the probability of the particular outcome (see diagram).

Here the probability of a six followed by a six is found by tracing the branch corresponding to this outcome through the tree.

Note that the ends of the branches of the tree are usually known as **terminal nodes**.

A more complicated example

A machine is used to produce components. Each time it produces a component there is a chance that the component will be defective.

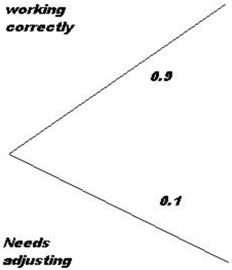
When the machine is working correctly the **probability that a component is defective is 0.05**. Sometimes, though, the machine requires adjustment and, when this is the case, the **probability that a component is defective is 0.2**.

At the time in question there is a **probability of 0.1 that the machine requires adjustment**. Components produced by the machine are tested and either accepted or rejected.

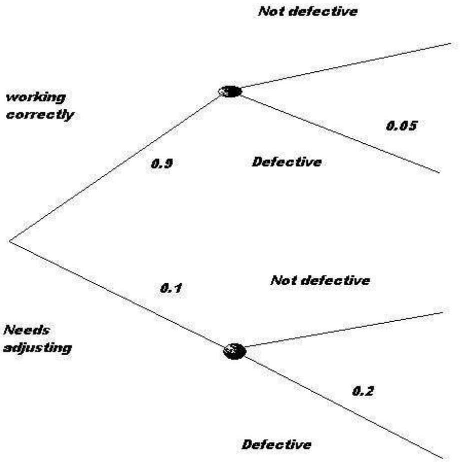
A component which is not defective is **accepted with probability 0.97** and (falsely) **rejected with probability 0.03**; A defective component is (falsely) **accepted with probability 0.15** and **rejected with probability 0.85**.

Construct a tree diagram to represent this scenario.

A more complicated example



A more complicated example



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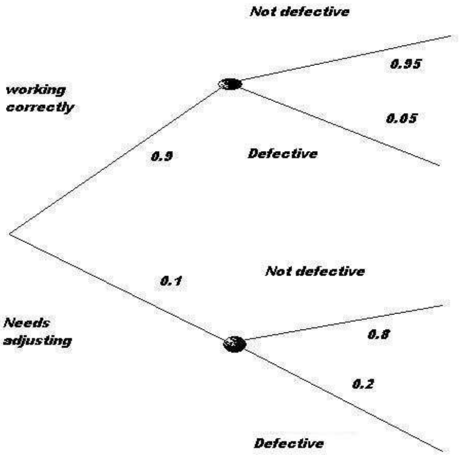

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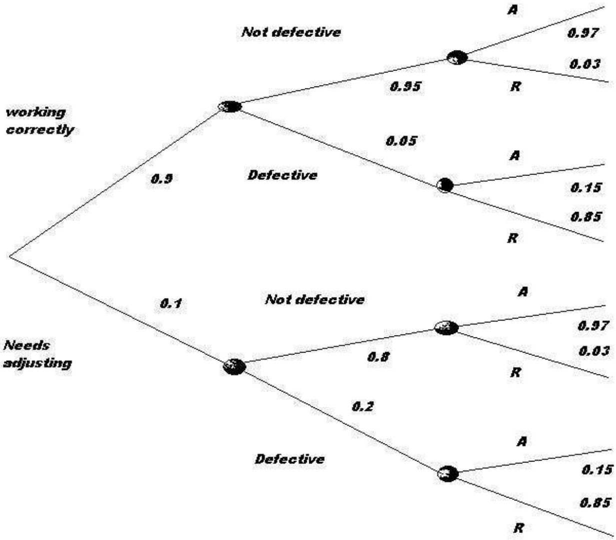

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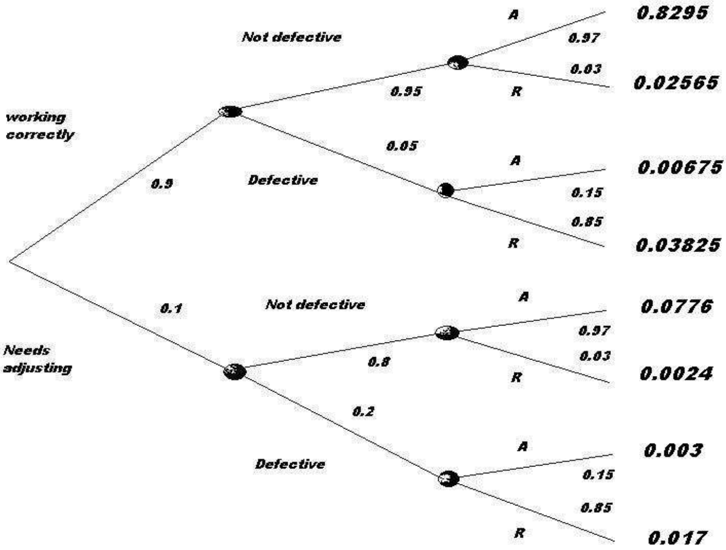

We can calculate various probabilities. For example:

$$
\begin{aligned}
P(\textbf{accepted}) &= 0.82935 + 0.00675 + 0.07760 + 0.00300 \\
&= 0.9167
\end{aligned}
$$

$$
\begin{aligned}
P(\textbf{defective}) &= (0.9 \times 0.05) + (0.1 \times 0.2) \\
&= 0.045 + 0.02 \\
&= 0.065
\end{aligned}
$$

$$
P(\textbf{defective and accepted}) = 0.00675 + 0.00300 = 0.00975
$$

$$
P(\textbf{accepted} \mid \textbf{defective}) = \frac{0.00975}{0.065} = 0.15
$$

$$\begin{aligned}
P(\textbf{defective} \mid \textbf{accepted}) &= \frac{0.00975}{0.9167} = 0.010636 \\
P(\textbf{machine OK and accepted}) &= 0.82935 + 0.00675 = 0.8361 \\
P(\textbf{machine OK} \mid \textbf{accepted}) &= \frac{0.8361}{0.9167} = 0.9121 \\
P(\textbf{machine OK and rejected}) &= 0.02565 + 0.03825 = 0.0639
\end{aligned}$$

$$\begin{aligned} P(\textbf{rejected}) &= 1 - P(\textbf{accepted}) = 0.0833 \\ P(\textbf{machine OK} \mid \textbf{rejected}) &= \frac{0.0639}{0.0833} = 0.7671 \end{aligned}$$