Solutions to Exercises 9

1. Let X be the length of a steel beam made in the steel mill. Then $X \sim N(8.25, 0.07^2)$, that is, X has a normal distribution with mean $\mu = 8.25$m and standard deviation $\sigma = 0.07$m.

(a) The probability that the length of a steel beam is less than 8.4m is $P(X < 8.4)$. To use the formula

$$P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

we need to calculate

$$z = \frac{8.4-\mu}{\sigma} = \frac{8.4-8.25}{0.07} = 2.14286$$

and from tables we obtain $P(Z < 2.14) = 0.9838$.

(b) The probability that the length of a steel beam is between 8.2m and 8.4m is

$$P(8.2 < X < 8.4) = P(X < 8.4) - P(X < 8.2).$$

From (a), using tables we have $P(X < 8.4) = 0.9838$. Also

$$z = \frac{8.2-\mu}{\sigma} = \frac{8.2-8.25}{0.07} = -0.714288$$

and from tables we obtain $P(Z < -0.71) = 0.2389$. Therefore

$$\begin{aligned} P(8.2 < X < 8.4) &= 0.9838 - 0.2389 \\ &= 0.7449. \end{aligned}$$

(c) The probability that the length of a steel beam is less than 8.05m is $P(X < 8.05)$. Now

$$z = \frac{8.05-\mu}{\sigma} = \frac{8.05-8.25}{0.07} = -2.85714$$

and from tables we obtain $P(Z < -2.86) = 0.0021$. Therefore

$$P(X < 8.05) = 0.0021 = 0.21\%.$$

(d) We need the length x so that $P(X < x) = 0.98$. First, we need the value of z so that $P(Z < z) = 0.98$. Using tables, the two key probabilities are

$$P(Z < 2.05) = 0.9798 \quad \text{and} \quad P(Z < 2.06) = 0.9803$$

and so we take $z = 2.05$.

In other words $P(Z < 2.05) \approx 0.98$. Moving back to the length scale, we need the value x such that $P(X < x) = 0.98$ and so we take

$$\begin{aligned} x &= \mu + z\sigma \\ &= 8.25 + 2.05 \times 0.07 \\ &= 8.394. \end{aligned}$$

2. The amount of liquid discharged X has a normal distribution with mean $\mu = 200$ml and standard deviation $\sigma = 15$ml.

(a) The probability that a cup contains less than 170ml is $P(X < 170)$. Now

$$z = \frac{170 - \mu}{\sigma} = \frac{170 - 200}{15} = -2$$

and from tables we obtain $P(Z < -2) = 0.0228$. Therefore

$$P(X < 170) = 0.0228 = 2.28\%.$$

(b) The probability that a cup contains over 225ml is $P(X > 225)$. Now $P(X > 225) = 1 - P(X \leq 225)$. Also

$$z = \frac{225 - \mu}{\sigma} = \frac{225 - 200}{15} = 1.6667$$

and from tables we obtain $P(Z < 1.67) = 0.9525$. Therefore

$$P(X > 225) = 1 - 0.9525 = 0.0475 = 4.75\%.$$

(c) The probability that the cup contains between 175ml and 225ml is

$$P(175 < X < 225) = P(X < 225) - P(X \leq 175).$$

From (a), using tables we have $P(X < 225) = 0.9525$. Also

$$z = \frac{175 - \mu}{\sigma} = \frac{175 - 200}{15} = -1.6667$$

and from tables we obtain $P(Z < -1.67) = 0.0475$. Therefore

$$\begin{aligned} P(175 < X < 225) &= 0.9525 - 0.0475 \\ &= 0.9050. \end{aligned}$$

(d) If a 240ml cup is used then it will overflow with probability $P(X > 240)$. Now $P(X > 240) = 1 - P(X \leq 240)$. Also

$$z = \frac{240 - \mu}{\sigma} = \frac{240 - 200}{15} = 2.6667$$

and from tables we obtain $P(Z < 2.67) = 0.9962$. Therefore

$$P(X > 240) = 1 - 0.9962 = 0.0038.$$

Hence, with 10000 drinks we would expect $10000 \times 0.0038 = 3.8$ to overflow.

3. Let X denote the time taken to deliver the goods. Then X has a normal distribution with mean $\mu = 16$ days and standard deviation $\sigma = 2.5$ days.

(a) The probability of a delivery being late is $P(X > 20)$. Now $P(X > 20) = 1 - P(X \leq 20)$. Also

$$z = \frac{20 - \mu}{\sigma} = \frac{20 - 16}{2.5} = 1.6$$

and from tables we obtain $P(Z < 1.6) = 0.9452$. Therefore

$$P(X > 20) = 1 - 0.9452 = 0.0548.$$

(b) The probability that customers receive their orders between 10 and 15 days is

$$P(10 < X < 15) = P(X < 15) - P(X \leq 10).$$

Now

$$z = \frac{15 - \mu}{\sigma} = \frac{15 - 16}{2.5} = -0.4$$

and from tables we obtain $P(Z < -0.4) = 0.3446$, and so $P(X < 15) = 0.3446$. Also

$$z = \frac{10 - \mu}{\sigma} = \frac{10 - 16}{2.5} = -2.4$$

and from tables we obtain $P(Z < -2.4) = 0.0082$, and so $P(X \leq 10) = 0.0082$. Therefore

$$\begin{aligned} P(10 < X < 15) &= 0.3446 - 0.0082 \\ &= 0.3364. \end{aligned}$$

(c) If only 3% of deliveries are late then we need the number of days x so that $P(X \leq x) = 0.97$. First, we need the value of z so that $P(Z < z) = 0.97$. Using tables, the two key probabilities are

$$P(Z < 1.88) = 0.9699 \quad \text{and} \quad P(Z < 1.89) = 0.9706$$

and so we take $z = 0.9699$.

In other words $P(Z < 1.88) \approx 0.97$. Moving back to the delivery time scale, we need the value x such that $P(X < x) = 0.97$ and so we take

$$\begin{aligned} x &= \mu + z\sigma \\ &= 16 + 1.88 \times 2.5 \\ &= 20.7 \text{ days.} \end{aligned}$$

(d) In the new processing system, X has a normal distribution with mean $\mu = 16$ days and standard deviation $\sigma = 1.5$ days. The probability of a delivery on time is $P(X < 20)$. Now

$$z = \frac{20 - \mu}{\sigma} = \frac{20 - 16}{1.5} = 2.6667$$

and from tables we obtain $P(Z < 2.67) = 0.9962$. Therefore

$$P(X < 20) = 0.9962.$$

This is an increase on the previous system of over 5%.