Solutions to Exercises 10

1. The amount of time (in minutes) that the coach is delayed X has a uniform distribution on $a = -15$ to $b = 45$.

(a) The pdf is a flat line, height $1/\{45 - (-15)\} = 1/60$, in the range -15 to 45. The pdf is zero everywhere else.

(b) The mean of this distribution is

$$E(X) = \frac{a+b}{2} = \frac{45 + (-15)}{2} = 15 \text{ minutes},$$

so that, on average, the coach is 15 minutes late. Also, the variance is

$$Var(X) = \frac{\{45 - (-15)\}^2}{12} = \frac{3600}{12} = 300$$

and therefore $SD(X) = \sqrt{Var(X)} = \sqrt{300} = 17.32$ minutes.

Probabilities for this distribution are calculated using

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b \end{cases}$$

$$= \begin{cases} 0 & \text{for } x < -15 \\ \dfrac{x+15}{60} & \text{for } -15 \leq x \leq 45 \\ 1 & \text{for } x > 45. \end{cases}$$

(c) The probability that the coach is less than 5 minutes late is

$$P(X < 5) = \frac{5+15}{60} = \frac{20}{60} = \frac{1}{3} = 0.3333.$$

(d) The probability that the coach is more than 20 minutes late is

$$P(X > 20) = 1 - P(X \leq 20) = 1 - \frac{20+15}{60} = 1 - \frac{35}{60} = \frac{5}{12} = 0.4167.$$

(e) The probability that the coach arrives between 22.55 and 23.20 is

$$\begin{aligned} P(-5 < X < 20) &= P(X < 20) - P(X \leq -5) \\ &= \frac{20+15}{60} - \frac{-5+15}{60} \\ &= \frac{7}{12} - \frac{1}{6} \\ &= \frac{5}{12} \\ &= 0.417. \end{aligned}$$

(f) The probability that the coach arrives at 23.00 depends on what is meant by "arrives at 23.00". If it has a strict meaning, that is, the coach arrives at exactly 23.00 (not even a billionth of a second out) then this event has probability zero. However, if this description means "to the nearest minute" then we need the probability that the coach arrives at 23.00 plus or minus half a minute. In terms of the random variable X, this probability is

$$\begin{aligned} P(-0.5 < X < 0.5) &= P(X < 0.5) - P(X \leq -0.5) \\ &= \frac{0.5 + 15}{60} - \frac{-0.5 + 15}{60} \\ &= \frac{15.5}{60} - \frac{14.5}{60} \\ &= \frac{1}{60} \\ &= 0.0167. \end{aligned}$$

(g) The question states that the coach cannot arrive more than 45 minutes late and so the probability it arrives at 0.00 is zero.

2. As the network server receives incoming requests according to a Poisson process with mean $\lambda = 2.5$ per minute, the time between successive requests X has an exponential distribution with parameter $\lambda = 2.5$ minutes.

(a) The expected time between arrivals of requests is

$$E(X) = \frac{1}{\lambda} = \frac{1}{2.5} = 0.4 \text{ minutes.}$$

Probabilities for this distribution are calculated using

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-2.5 \times x} & \text{for } x > 0. \end{cases}$$

(b) The probability that the time between requests is less than 2 minutes is

$$P(X < 2) = 1 - e^{-2.5 \times 2} = 1 - e^{-5} = 1 - 0.0067 = 0.9933.$$

(c) The probability that the time between requests is greater than 1 minute is

$$P(X > 1) = 1 - P(X < 1) = 1 - \left(1 - e^{-2.5 \times 1}\right) = e^{-2.5} = 0.0821.$$

(d) The probability that the time between requests is between 30 seconds and 50 seconds is

$$\begin{aligned}
P\left(\frac{1}{2} < X < \frac{5}{6}\right) &= P\left(X < \frac{5}{6}\right) - P\left(X \leq \frac{1}{2}\right) \\
&= 1 - e^{-2.5\times 5/6} - \left(1 - e^{-2.5\times 1/2}\right) \\
&= e^{-1.25} - e^{-12.5/6} \\
&= 0.2865 - 0.1245 \\
&= 0.1620.
\end{aligned}$$

3. Let X denote the time to first breakdown. Then

Company 1: X has an exponential distribution with $\lambda = 0.11$. Therefore, the probability of no breakdown within the first six months is

$$P(X > 6) = 1 - P(X < 6) = 1 - \left(1 - e^{-0.11\times 6}\right) = e^{-0.66} = 0.5169.$$

Company 2: X has an exponential distribution with $\lambda = 0.01$. Therefore, the probability of no breakdown within the first six months is

$$P(X > 6) = 1 - P(X < 6) = 1 - \left(1 - e^{-0.01\times 6}\right) = e^{-0.06} = 0.9418.$$

Recommend buy from Company 2 as their probability of no breakdown within the first six months is much larger than that of Company 1.