# Lecture 9

# LINEAR PROGRAMMING (I)

# Announcements

Welcome back! I hope you enjoyed the Easter break...

... now for the last leg of the course – the end is in sight!

Where we are: Half–way through the last topic in the course: **Statistical Modelling**.

- Correlation and regression
- Time series and forecasting
- Linear programming

# Announcements

**Week 9 (this week)**

- Linear programming I
- Assignment 2 due in Wednesday
- Tutorials as normal

**Week 10 (next week)**

- Linear programming II
- CBA6 opens in practice mode
- Tutorials as normal

# Announcements

**Week 11**

- Revision (semester 2 revision booklet)
- CBA6 opens in exam mode
- Tutorials as normal

**Week 12**

- Revision (specimen exam paper, hints and tips for exam!)
- Tutorials optional (but will be very helpful!)

*Exam the week after that!*

# Introduction

**Decision making** is a process that is carried out in many areas of life.

Usually there is a particular aim in making one decision rather than another.

Two aims often considered in business are:

- **maximising profit**, and
- **minimising cost**.

During World War Two, American mathematicians developed some mathematical methods to help the decision making process.

Their aims were to express all

- **requirements**
- **constraints** and
- **objectives**

as algebraic equations. They then developed methods for obtaining the **optimal solution** to the problem posed.

One such method is called **linear programming**.

Linear programming belongs to a field of statistics known as **operational research**.

For our set of algebraic equations to reflect the **requirements**, **constraints** and **objectives** of a real–life situation, you can imagine how complex they would be!

In this chapter, we will study simple problems, for which all the algebraic expressions are **linear**, i.e.

$$\textbf{(a number)}x + \textbf{(a number)}y \quad = \quad \textbf{a number}.$$

For example, we might express **profit** as a linear combination of two other variables:

$$4x + 3y \quad = \quad \text{Profit}.$$

This is a linear **equation**.

If we want our profit to be *at least* £50, we might consider the following linear **inequality**:

$$4x + 3y \geq £50.$$

In today's lecture, we will consider how to **formulate** real–life situations as linear programming problems.

Next week, we will discuss how to **solve** such problems.

# Formulating the problem

1. Identify the **decision variables**
   These are the quantities you need to know in order to solve the problem.
2. Identify the **constraints**
   For example, there may be a limit on resources or a maximum/minimum value a decision variable can take.
3. Determine the **objective function**
   This is the quantity to be be *optimised*, usually **profit** or **costs**.

We will consider three real–life examples:

- A **chair manufacturer**,
- A **book publisher**, and
- A **haulage company**.

# Example 1: A chair manufacturer

A manufacturer makes two kinds of chairs – **A** and **B**. Each type of chair has to be processed in two departments – $\mathbf{I}$ and $\mathbf{II}$.

Chair **A** spends 3 hours in department $\mathbf{I}$ and 2 hours in department $\mathbf{II}$. Chair **B** spends 3 hours in department $\mathbf{I}$ and 4 hours in department $\mathbf{II}$.

The time available in department $\mathbf{I}$ in any given month is 120 hours, and the time available in department $\mathbf{II}$ in the same month is 150 hours.

Chair **A** has a selling price of £10 and chair **B** of £12.

The manufacturer wishes to maximise his income.

**How many of each type of chair should be made?**

You'll notice that there's a lot of information given in the question – this is typical of a linear programming problem. Sometimes it's easier to summarise the information given in a table:

| Chair | Time in dept. I | Time in dept. II | Selling price |
|---|---|---|---|
| **A** | 3 | 2 | 10 |
| **B** | 3 | 4 | 12 |
| Time limits | 120 | 150 | |

To formulate this linear programming problem, we consider the following three steps:

1. What are the **decision variables**? (i.e. which quantities do you need to know in order to solve the problem?)
2. What are the **constraints**?
3. What is the **objective**?

# Step 1: Decision variables

Read through the question and identify the things you'd like to know. You can usually do this by going straight to the last sentence of the question:

**"How many of each chair should be made..."**

Thus, we'd like to know

- the number of type **A** chairs to make, and
- the number of type **B** chairs to make.

These are our decision variables, and are usually denoted with lower case letters. Thus, our decision variables are

$$\begin{aligned} x &= \text{number of type } \textbf{A} \text{ chairs made} \qquad \text{and} \\ y &= \text{number of type } \textbf{B} \text{ chairs made.} \end{aligned}$$

## Step 2: Constraints

This is probably the hardest bit! Consider what could happen in each department.

For example, if we focus on what could happen in department $\mathbf{I}$:

| | |
|---|---|
| Since: | the production of 1 type **A** chair uses 3 hours, |
| then: | the production of $x$ type **A** chairs takes $3x$ hours. |
| Similarly: | the production of 1 type **B** chair uses 3 hours, |
| so: | the production of $y$ type **B** chairs takes $3y$ hours. |

The total time used in department $\mathbf{I}$ is therefore

$$(3x + 3y) \text{ hours.}$$

Since only 120 hours are available in department $\mathbf{I}$, one constraint is

$$\begin{aligned} (3x + 3y) \text{ hours} &\leq 120 \text{ hours}, \qquad \text{or just} \\ (3x + 3y) &\leq 120. \end{aligned}$$

Considering department **II** in a similar way, we get:

| | |
|---|---|
| Since: | the production of 1 type **A** chair uses 2 hours, |
| then: | the production of $x$ type **A** chairs takes $2x$ hours. |
| | |
| Similarly: | the production of 1 type **B** chair uses 4 hours, |
| so: | the production of $y$ type **B** chairs takes $4y$ hours. |

The total time used is therefore

$$( 2x + 4y) \text{ hours.}$$

Since only 150 hours are available in department $\mathbf{II}$, a second constraint is

$$\begin{aligned} (2x + 4y) \text{ hours} &\leq 150 \text{ hours}, \qquad \text{or just} \\ (2x + 4y) &\leq 150. \end{aligned}$$

We're still not done! We can't make a negative number of chairs, so we also have:

$$\begin{aligned} x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

These are called the **non–negativity constraints**.

# Step 3: Objective function

Our objective here is to **maximise income**.

If we make $x$ type **A** chairs, then we get $£10 \times x = £10x$, since each type **A** chair sells for £10.

Similarly, if we make $y$ type **B** chairs, then we get $£12 \times y = £12y$, since each type **B** chair sells for £12.

The total income is then

$$\pounds Z = \pounds(10x + 12y).$$

The aim is to maximise income, so we'd like to maximise

$$Z = 10x + 12y,$$

where $Z$ is the objective function.

**Thus, to summarise, we have the following linear programming problem:**

Maximise $Z = 10x + 12y$ subject to the constraints

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

# Example 2: A book publisher

A book publisher is planning to produce a book in two different bindings: paperback and library. Each book goes through a sewing process and a gluing process. The table below gives the time required, in minutes, for each process and for each of the bindings:

| | Sewing (mins) | Gluing (mins) |
|---|---|---|
| Paperback | 2 | 4 |
| Library | 3 | 10 |

The sewing process is available for 7 hours per day and the gluing process for 15 hours per day.

The profits are 25p on a paperback edition and 60p on a library edition.

**How many books in each binding should be manufactured to maximise profits?**

It might be a good idea to extend this table to include all the information given by adding the restrictions on time and profits, i.e.

| | Sewing (mins) | Gluing | Profit ($P$) |
|---|---|---|---|
| Paperback | 2 | 4 | 25 |
| Library | 3 | 10 | 60 |
| Total time | 420 (in minutes!) | 900 (in minutes!) | |

# Step 1: Decision variables

The decision variables are the number of books to be made in each binding. Let

$$\begin{aligned} x &= \text{number in paperback binding} \qquad \text{and} \\ y &= \text{number in library binding.} \end{aligned}$$

## Step 2: Constraints

The constraints are:

$$\begin{aligned} \textbf{sewing} : 2x + 3y &\leq 420 \qquad \text{and} \\ \textbf{gluing} : 4x + 10y &\leq 900, \end{aligned}$$

together with the non–negativity conditions

$$\begin{aligned} x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

# Step 3: Objective function

For each paperback edition, the publisher makes 25p profit. Since we make $x$ number of paperback bindings, the publisher will make $25x$ pence profit.

Similarly, for each library edition, the publisher makes 60p profit. Since we make $y$ number of library bindings, the publisher will make $60y$ pence profit.

The objective is to maximise the profit $P$ pence. The total profit is $25x + 60y$ – thus our aim is to maximise

$$P = 25x + 60y.$$

**Thus, to summarise, we have the following linear programming problem:**

Maximise $P = 25x + 60y$ subject to the constraints

$$\begin{aligned} 2x + 3y &\leq 420, \\ 4x + 10y &\leq 900, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$