Lecture 10

LINEAR PROGRAMMING (II)

Graphical solutions for two–variable problems

In last week's lecture we discussed how to **formulate** a linear programming problem.

This week, we consider how to **solve** such problems.

Linear programming problems that involve only two decision variables – say x and y – may be solved **graphically** or **algebraically**.

Sets of points defined by a linear inequality

Any equation of the form

$$ax + by = c,$$

where a, b and c are numbers, is called a **linear equation**.

For example,

$$3x + 4y = 12$$

is a linear equation.

In the x, y plane this is the equation of a straight line, and this line may be drawn by identifying any two points on it.

Example

Suppose we have 3x + 4y = 12.

We can identify two points which lie on the straight line by considering what happens when $x = 0$ and what happens when $y = 0$.

- When $x = 0$, we have

$$\begin{aligned} 3 \times \mathbf{0} + 4y &= 12 \qquad \text{i.e.} \\ 0 + 4y &= 12 \qquad \text{i.e.} \\ 4y &= 12 \qquad \text{i.e.} \\ y &= 3. \end{aligned}$$

  So one point on the line $3x + 4y = 12$ is at $x = 0, y = 3$, or $(0, 3)$.

- When $y = 0$, we have

$$\begin{aligned} 3x + 4 \times \mathbf{0} &= 12 \qquad \text{i.e.} \\ 3x + 0 &= 12 \qquad \text{i.e.} \\ 3x &= 12 \qquad \text{i.e.} \\ x &= 4. \end{aligned}$$

  So another point on the line $3x + 4y = 12$ is at $x = 4, y = 0$, or $(4, 0)$.

Since we have the coordinates of two points which lie on the line, and the line is a straight line, we can plot the line with this equation! (figure 10.1)

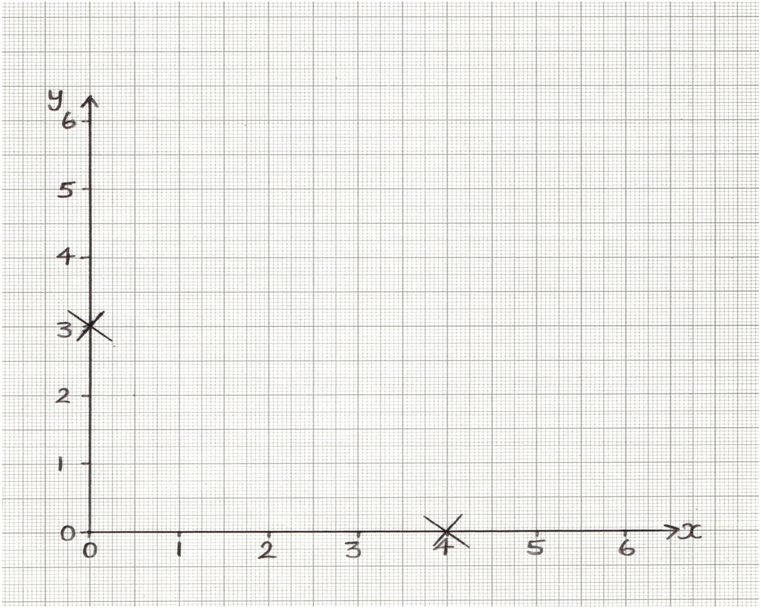

Figure: **10.1**: Plot of the line $3x + 4y = 12$

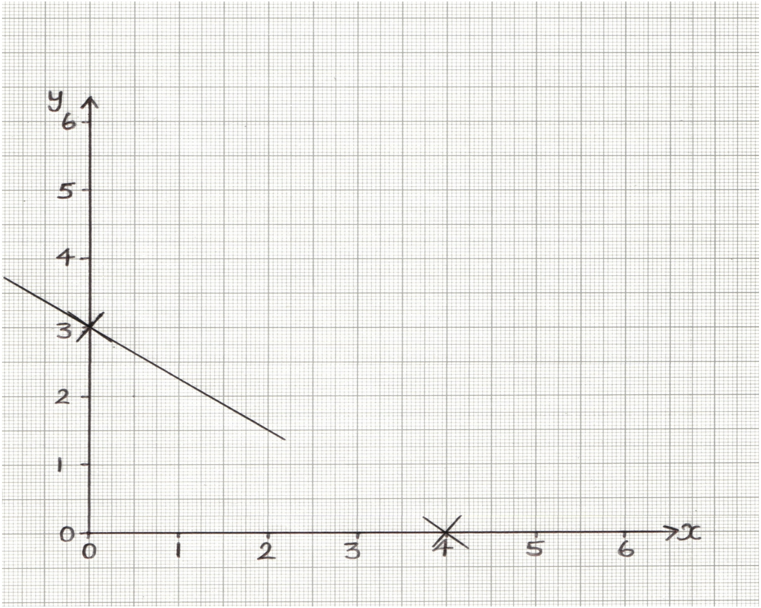

Figure: **10.1**: Plot of the line $3x + 4y = 12$

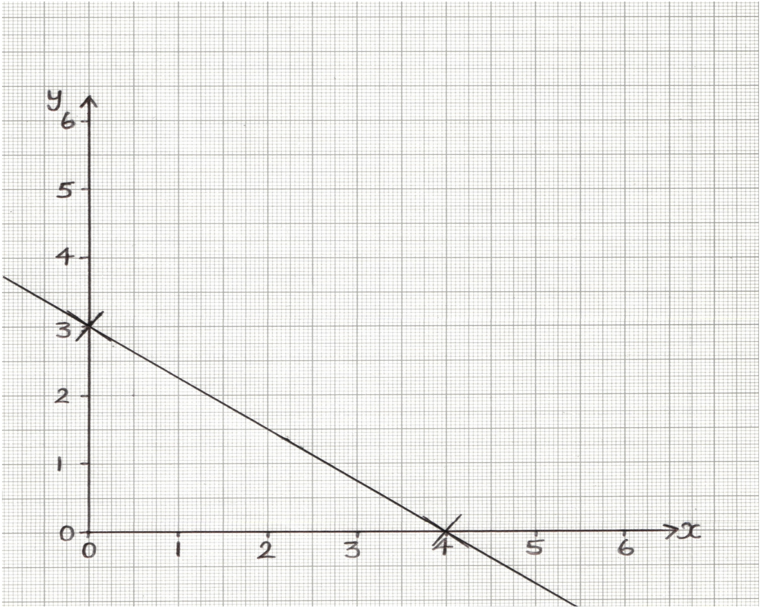

Figure: **10.1**: Plot of the line $3x + 4y = 12$

So how do we show a linear **inequality** on a graph?

Any straight line divides the xy–plane into two half–planes.

If the equation of the line is $ax + by = c$ then

- on one side of the line you have $ax + by < c$
- on the other side of the line you have $ax + by > c$.

Suppose now we have $3x + 4y \leq 12$.

1. Draw the line $3x + 4y = 12$
2. We want the region which is *less than or equal to 12*
   - The line on our graph splits the xy–plane into two regions...
   - ... which one do we choose?
   - Use the Origin to find out!
3. Shade out the region you don't want to leave the **admissible set** of points defined by the inequality.

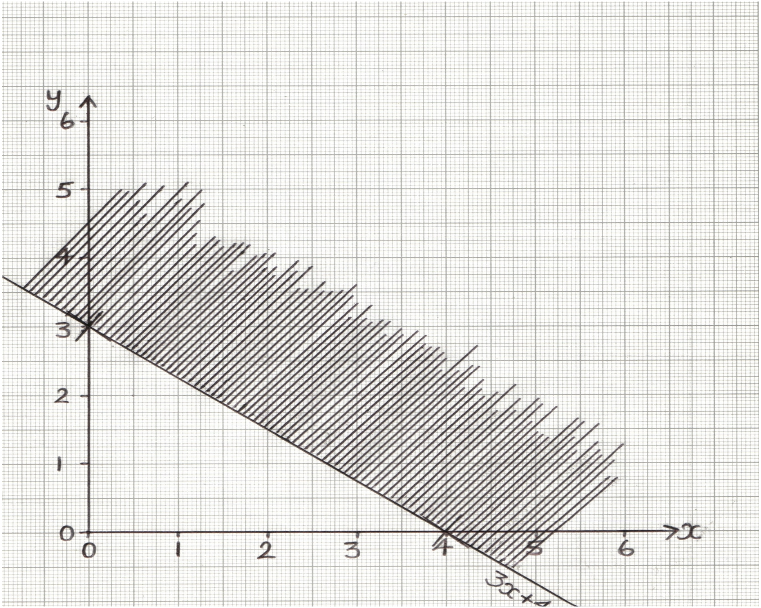

Figure: **10.2**: Indicating the region for which $3x + 4y \leq 12$

Sets of points defined by a collection of inequalities

Each inequality in a linear programming problem will produce an **admissible set**.

To find a solution to the problem you need to find the set of points which satisfies all the inequalities **simultaneously**.

This is obtained graphically by drawing a diagram like the one in figure 10.2 but showing **all** the inequalities.

The required region, or feasible region, is then the one which does not contain any shading marks.

Example

Indicate on a diagram the region for which

$$\begin{aligned} 3x + 4y &\leq 12, \\ 3x + 2y &\leq 9, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

We've already looked at how to show the admissible set for the first inequality $3x + 4y \leq 12$.

The same procedure can be used for the inequality $3x + 2y \leq 9$.

Firstly, we need to plot the line $3x + 2y = 9$. Again, it's easiest to consider what happens when $x = 0$ and $y = 0$. For example,

- When $x = 0$, we have

$$\begin{aligned} 3 \times \mathbf{0} + 2y &= 9 \quad \text{i.e.} \\ 0 + 2y &= 9 \quad \text{i.e.} \\ 2y &= 9 \quad \text{so} \\ y &= 4.5. \end{aligned}$$

- Similarly, when $y = 0$, we have

$$\begin{aligned} 3x + 2 \times \mathbf{0} &= 9 \quad \text{i.e.} \\ 3x + 0 &= 9 \quad \text{i.e.} \\ 3x &= 9 \quad \text{so} \\ x &= 3. \end{aligned}$$

So the points $x = 0, y = 4.5$ and $x = 3, y = 0$, i.e. $(0, 4.5)$ and $(3, 0)$ lie on the line with equation $3x + 2y = 9$.

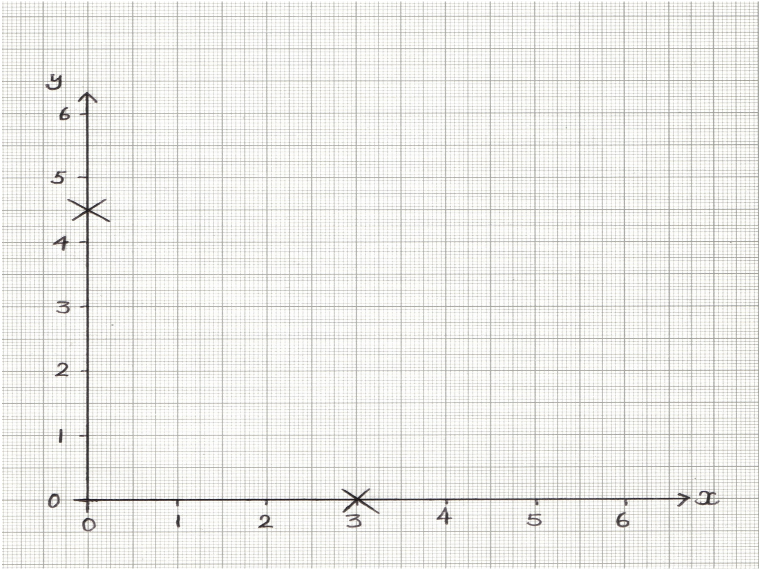

Figure: **10.3**: Plot of the line $3x + 2y = 9$; also shown is the admissible region for the inequality $3x + 2y \leq 9$

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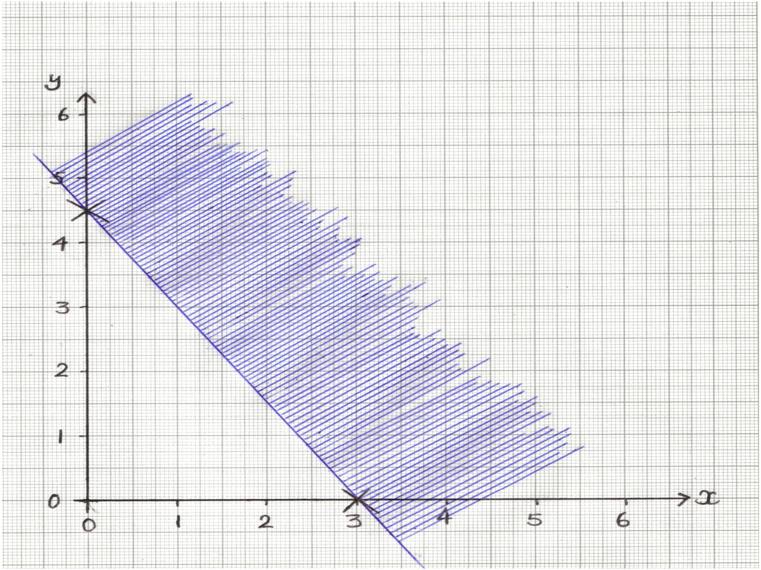

Figure: **10.3**: Plot of the line $3x + 2y = 9$; also shown is the admissible region for the inequality $3x + 2y \leq 9$

The inequalities $x \geq 0$ and $y \geq 0$ give:

Combining these with the inequality $3x + 4y \leq 12$ considered previously, we get

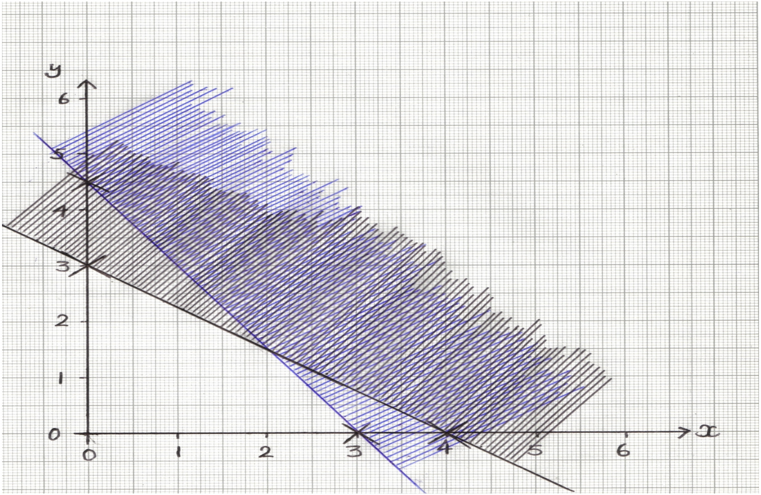

Figure: **10.4**: Diagram showing the admissible region satisfying the inequalities $3x + 4y \leq 12$, $3x + 2y \leq 9$, $x \geq 0$ and $y \geq 0$

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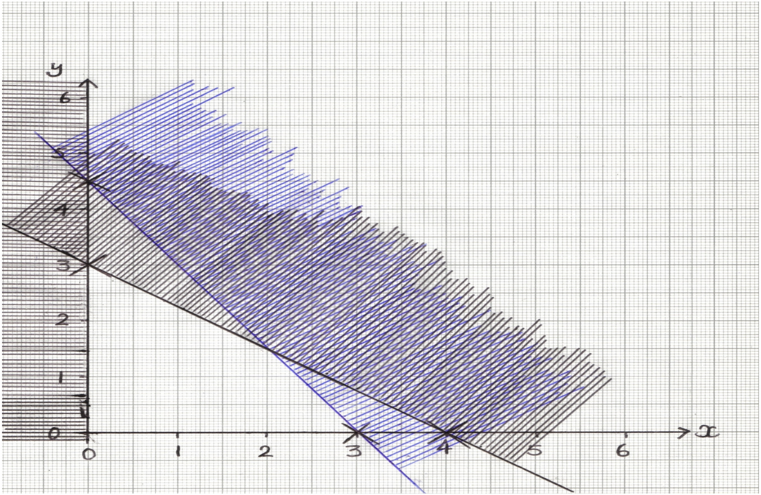

Figure: **10.4**: Diagram showing the admissible region satisfying the inequalities $3x + 4y \leq 12$, $3x + 2y \leq 9$, $x \geq 0$ and $y \geq 0$

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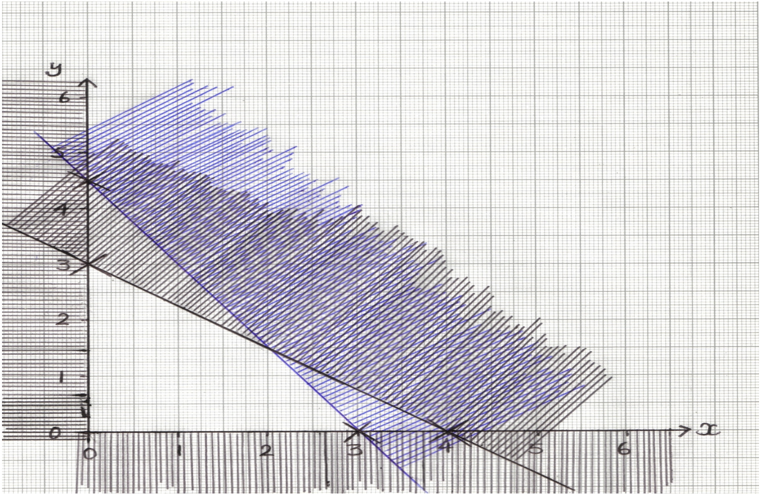

Figure: **10.4**: Diagram showing the admissible region satisfying the inequalities $3x + 4y \leq 12$, $3x + 2y \leq 9$, $x \geq 0$ and $y \geq 0$

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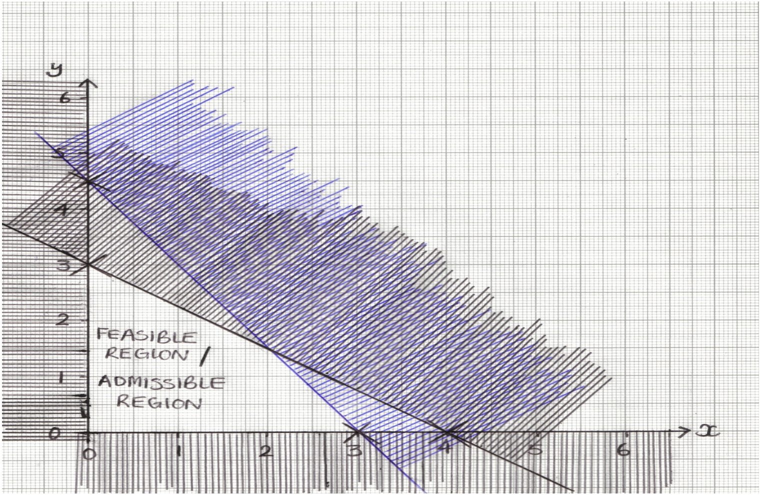


Figure: **10.4**: Diagram showing the admissible region satisfying the inequalities $3x + 4y \leq 12$, $3x + 2y \leq 9$, $x \geq 0$ and $y \geq 0$

We now use these techniques to obtain feasible solutions to the linear programming problems discussed last week, and look at how to find the **optimal** solution for each of these problems.

Recall the three scenarios:

1. a chair manufacturer,
2. a book publisher, and
3. a haulage company.

The chair manufacturer

Recall the problem posed in **example 9.2.1**:

A manufacturer makes two kinds of chairs, **A** and **B**, each of which has to be processed in two departments, I and II.

Chair **A** has to be processed in department I for 3 hours and in department II for 2 hours. Chair **B** has to be processed in department I for 3 hours and in department II for 4 hours.

The time available in department I in any given month is 120 hours, and the time available in department II, in the same month, is 150 hours.

Chair **A** has a selling price of £10 and chair **B** has a selling price of £12.

The manufacturer wishes to maximise his income. How many of each chair should be made in order to achieve this objective?

Last week we considered how to **formulate** this situation as a linear programming problem.

Using the graphical techniques outlined in this lecture, we will now consider how to **solve** this problem.

The table below summarises the information given in this particular problem.

Type of chair	Dept. I (hours)	Dept. II (hours)	Price (£)
A	3	2	10
B	3	4	12
Time available	**120**	**150**	

Recall the three steps involved in formulating a linear programming problem:

1. Identify the **decision variables**
2. Identify the **constraints**
3. State the **objective function**

In this example, the **decision variables** were identified as

$$\begin{aligned} x &= \text{number of type A chairs made} \qquad \text{and} \\ y &= \text{number of type B chairs made.} \end{aligned}$$

The **constraints** were

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

We then identified that the **objective** was to maximise the income, which we called Z, where

$$Z = 10x + 12y.$$

So, in summary, the linear programming problem is:

Maximise $Z = 10x + 12y$ subject to the following constraints:

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

The first inequality is $3x + 3y \leq 120$. To show this on a diagram, we first need to plot the line $3x + 3y = 120$.

- When $x = 0$, we have

$$\begin{aligned} 3 \times 0 + 3y &= 120 \qquad \text{i.e.} \\ 3y &= 120 \qquad \text{i.e.} \\ y &= 40. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 3x + 3 \times 0 &= 120 \qquad \text{i.e.} \\ 3x &= 120 \qquad \text{i.e.} \\ x &= 40. \end{aligned}$$

These points should be plotted on Figure 10.5, and the line with equation $3x + 3y = 120$ drawn.

Since we want $3x + 3y \leq 120$, our region of interest lies on or below the line, and so we shade out the space above the line.

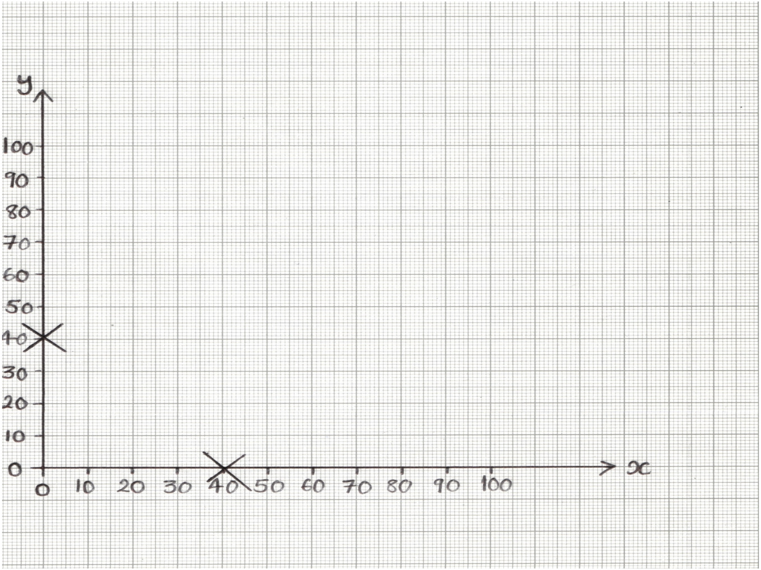
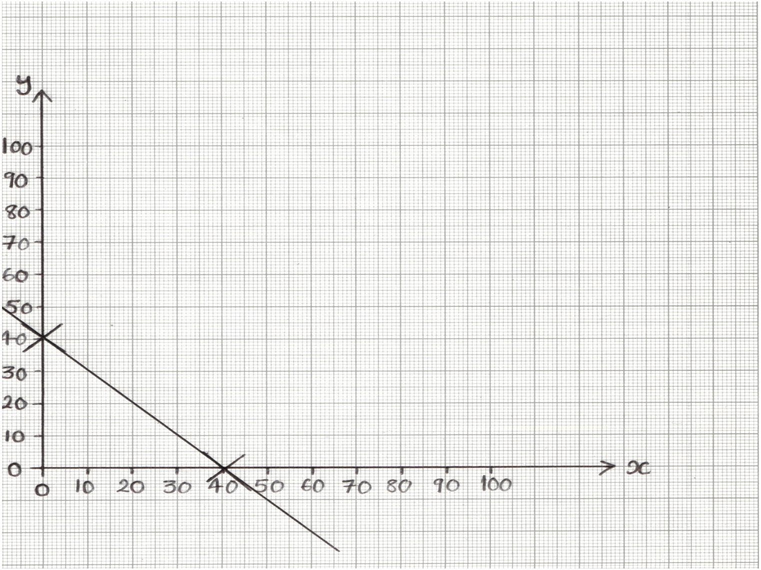


Figure: **10.5**: Feasible region and objective function for the chair manufacturing problem



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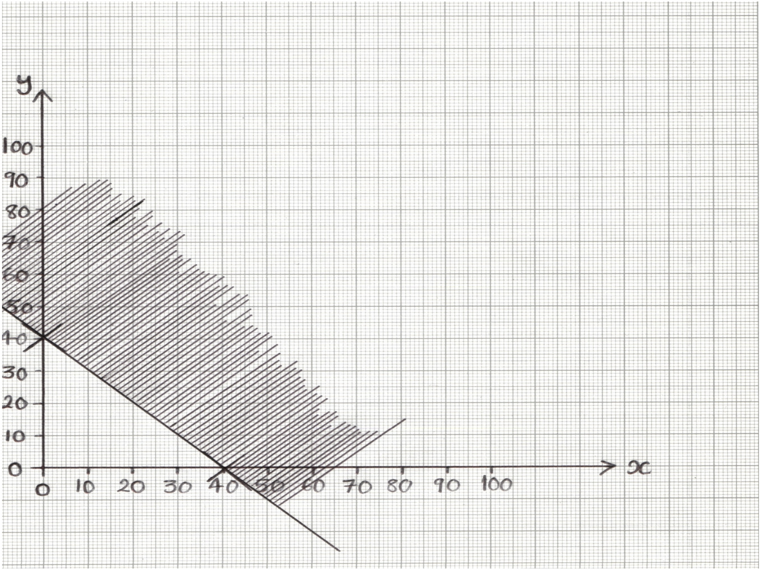


Figure: **10.5**: Feasible region and objective function for the chair manufacturing problem

Now consider the second inequality $2x + 4y \leq 150$. Again, to show this on a diagram, we first need to plot the line $2x + 4y = 150$.

- When $x = 0$, we have

$$\begin{aligned} 2 \times 0 + 4y &= 150 \quad \text{i.e.} \\ 4y &= 150 \quad \text{i.e.} \\ y &= 37.5. \end{aligned}$$

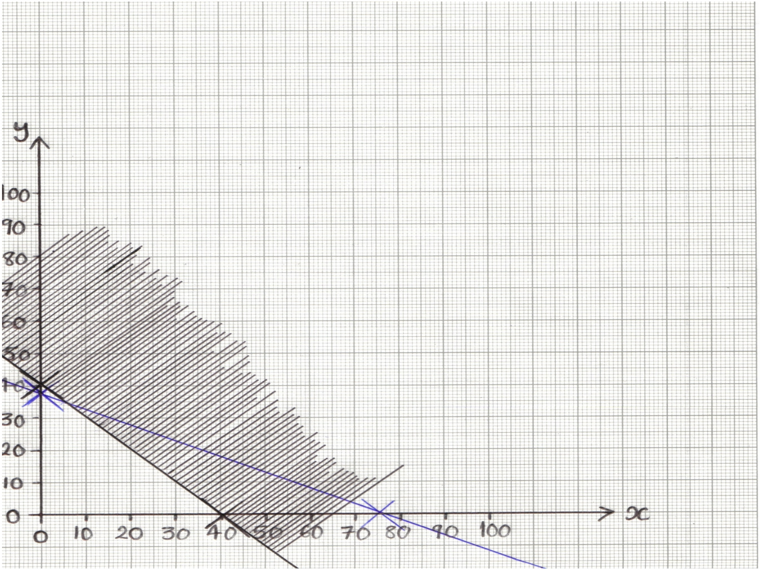
- When $y = 0$, we have

$$\begin{aligned} 2x + 4 \times 0 &= 150 \quad \text{i.e.} \\ 2x &= 150 \quad \text{i.e.} \\ x &= 75. \end{aligned}$$

Again, these points should be plotted on figure 10.5, and the line with equation $2x + 4y = 150$ drawn.

Since we want $2x + 4y \leq 150$, our region of interest lies on or below the line, and so we shade out the space above the line.

On figure 10.5 we also shade out the inadmissible regions for the two **non–negativity constraints**.



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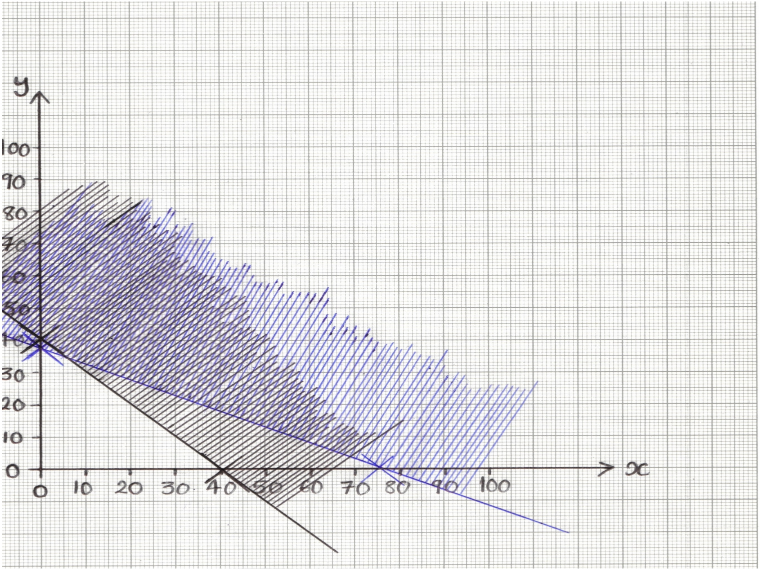

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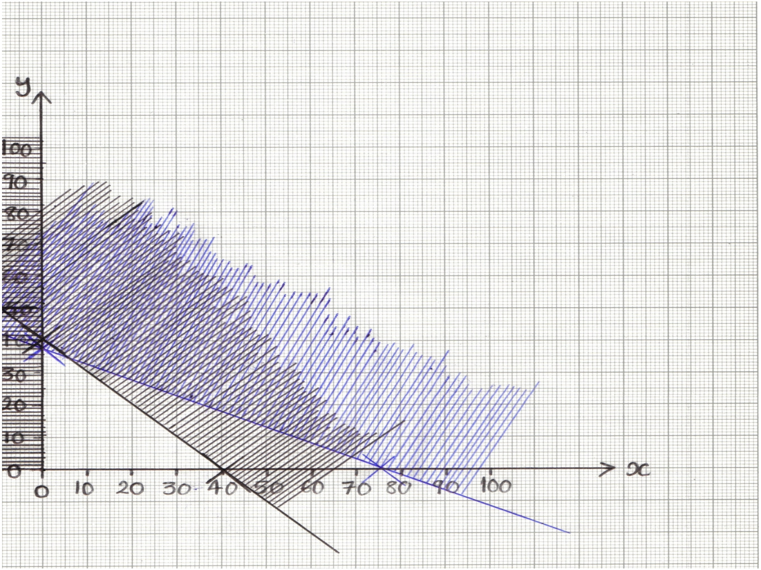

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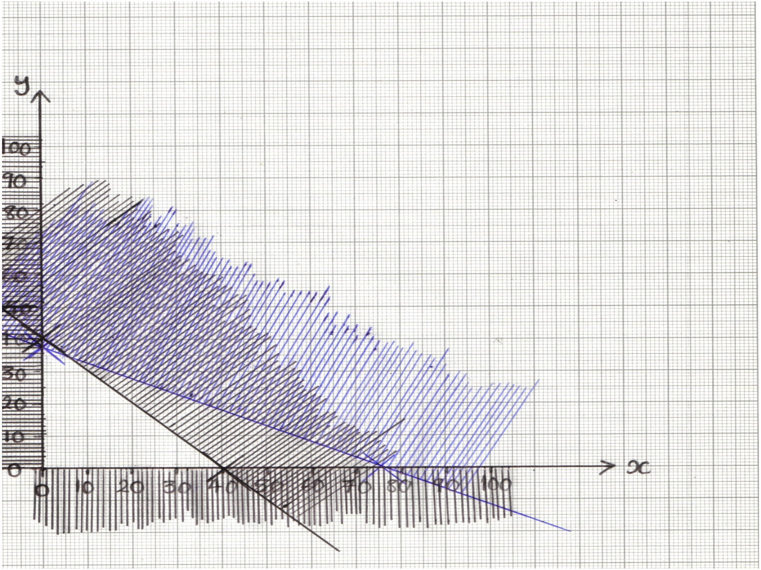


Figure: **10.5**: Feasible region and objective function for the chair manufacturing problem

The unshaded region in figure 10.5 shows the **feasible region** associated with our set of inequalities.

What we must do now is find the point in that region which meets our objective – i.e. the point in that region which **maximises income**.

One way of doing this is to also plot the objective function.

Our objective function is

$$Z = 10x + 12y,$$

where Z is our income.

- When Z takes different values we get a family of **parallel straight lines**
- We need to choose a **starting value** for Z in order to be able to plot the objective function
- It's often a good idea to try a value which is a multiple of both the coefficients of x and y...
- ... the coefficient of x is 10 and the coefficient of y is 12, so we could try a starting value of $Z = 120$.

The objective function is now

$$10x + 12y = 120.$$

We can plot this line in the same way as before – i.e. consider what happens when x and y are zero.

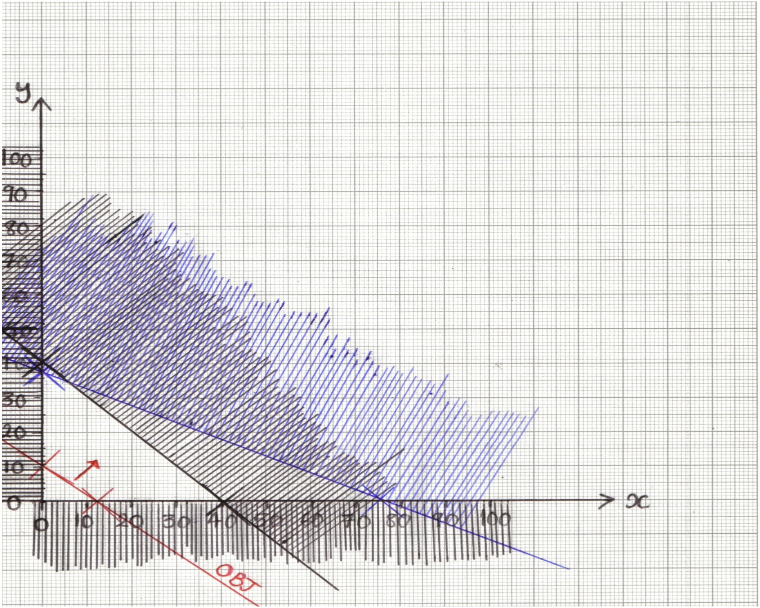
- When $x = 0$, we have

$$\begin{aligned} 10 \times \mathbf{0} + 12y &= 120 \qquad \text{i.e.} \\ 12y &= 120 \qquad \text{i.e.} \\ y &= 10. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 10x + 12 \times \mathbf{0} &= 120 \qquad \text{i.e.} \\ 10x &= 120 \qquad \text{i.e.} \\ x &= 12. \end{aligned}$$

This line should also be plotted on figure 10.5:



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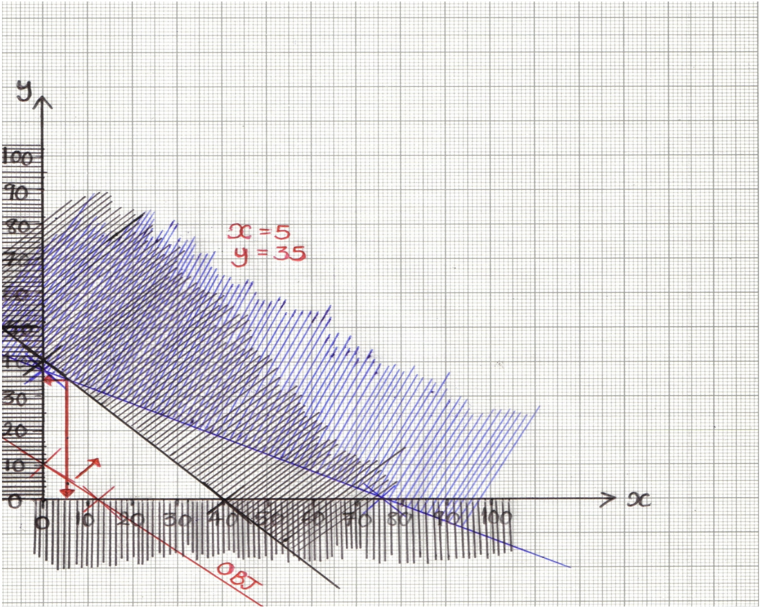

Notice that this line does not give the **optimal income**.

The origin represents **zero income**, and we want to move as far away from this as possible.

The largest value of Z (income) will occur at the point in the feasible region that is *furthest* from the origin, but still *parallel* to the objective line.

All points in the feasible region satisfy our inequalities, but only **one** point **maximises** income.

Once this point has been identified, we can simply "read off" the x and y values.

Doing so give $x = 5$ and $y = 35$, and so, in order to maximise income, we should make **5** type A chairs and **35** type B chairs.

This will give an income of

$$
\begin{aligned}
Z &= 10x + 12y \qquad \text{i.e.} \\
Z &= 10 \times \mathbf{5} + 12 \times \mathbf{35} \qquad \text{i.e.} \\
Z &= 50 + 420 \\
&= 470,
\end{aligned}
$$

i.e. £470.

The book publisher

Recall the problem posed in **example 9.2.2**:

A book publisher is planning to produce a book in two different bindings: paperback and library.

Each book goes through a sewing process and a gluing process. The sewing process is available for 7 hours per day and the gluing process for 15 hours per day.

The profits are 25p on a paperback edition and 60p on a library edition.

How many books in each binding should be manufactured to maximise profits? (assume that the publisher sells as many of each type of book as is produced.)

The table below summarises the information given in the above paragraph:

	Sewing (mins)	Gluing	Profit (P)
Paperback	2	4	25
Library	3	10	60
Total time	**420**	**900**	

The linear programming problem is summarised below:

Maximise $P = 25x + 60y$ subject to the constraints

$$\begin{aligned} 2x + 3y &\leq 420, \\ 4x + 10y &\leq 900, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

For the inequality $2x + 3y \leq 420$:

- When $x = 0$, we have

$$\begin{aligned} 2 \times 0 + 3y &= 420 \qquad \text{i.e.} \\ 3y &= 420 \qquad \text{i.e.} \\ y &= 140. \end{aligned}$$

- Similarly, when $y = 0$, we have

$$\begin{aligned} 2x + 3 \times 0 &= 420 \qquad \text{i.e.} \\ 2x &= 420 \qquad \text{i.e.} \\ x &= 210. \end{aligned}$$

We consider the inequality $4x + 10y \leq 900$ in a similar way:

- When $x = 0$, we have

$$\begin{aligned} 4 \times 0 + 10y &= 900 \qquad \text{i.e.} \\ 10y &= 900 \qquad \text{i.e.} \\ y &= 90. \end{aligned}$$

- Similarly, when $y = 0$, we have

$$\begin{aligned} 4x + 10 \times 0 &= 900 \qquad \text{i.e.} \\ 4x &= 900 \qquad \text{i.e.} \\ x &= 225. \end{aligned}$$

Both these lines should be plotted on figure 10.6:

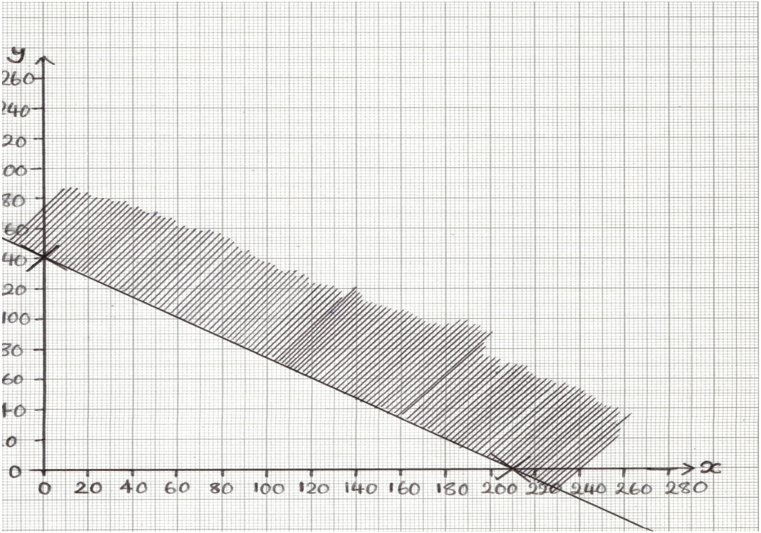

Figure: **10.6**: Feasible region and objective function for the book publisher's problem

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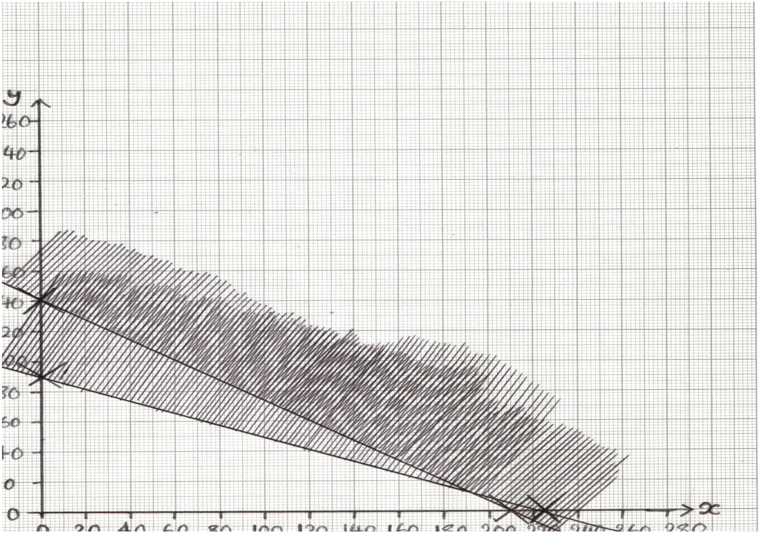

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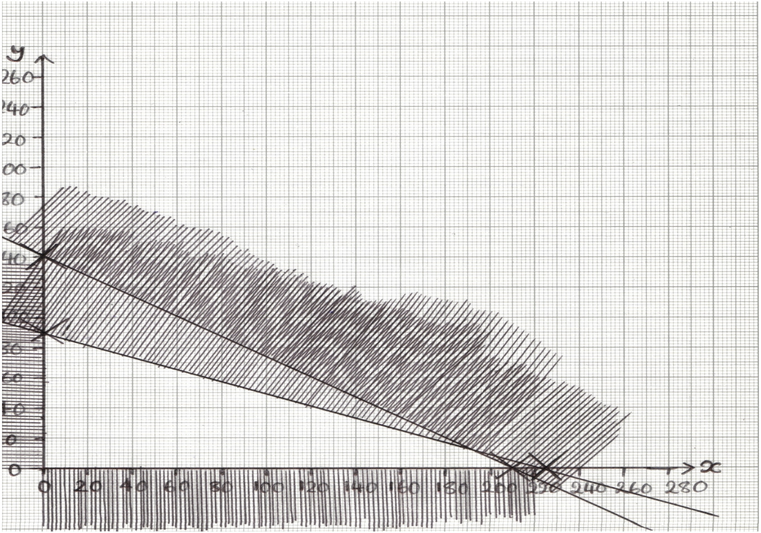


Figure: **10.6**: Feasible region and objective function for the book publisher's problem

We now need to plot the **objective function**.

The objective is to maximise profit, P, where

$$P = 25x + 60y.$$

Try $P = 25 \times 60 = 1500$.

To plot this line, we consider what happens when x and y are zero.

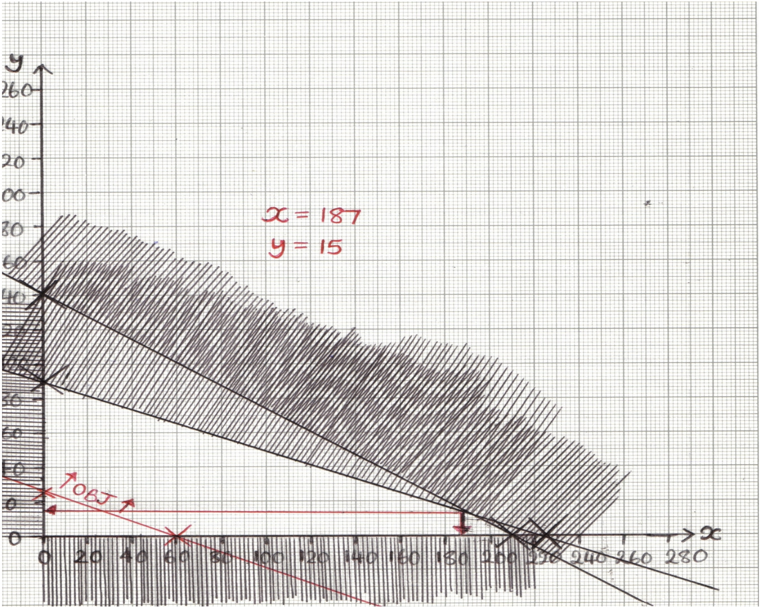
- When $x = 0$, we have

$$\begin{aligned} 25 \times 0 + 60y &= 1500 \qquad \text{i.e.} \\ 60y &= 1500 \qquad \text{i.e.} \\ y &= 25. \end{aligned}$$

- Similarly, when $y = 0$ we have

$$\begin{aligned} 25x + 60 \times 0 &= 1500 \qquad \text{i.e.} \\ 25x &= 1500 \qquad \text{i.e.} \\ x &= 60. \end{aligned}$$

This line should also be plotted in figure 10.6:



As before, we need to move this line as far away from the origin as possible in order to maximise profits, but keep the line inside the feasible region.

The point in the feasible region which is furthest away from the origin, but parallel to the objective line, is the intersection between the lines with equations $2x + 3y = 420$ and $4x + 10y = 900$.

As before, we can "read off" our solutions from the graph.

Doing so gives $x = 187.5$ and $y = 15$.

Thus, the publisher should make 187 paperback bindings and 15 library bindings in order to maximise profit. This will give

$$\begin{aligned} P &= 25x + 60y \\ &= 25 \times 187 + 60 \times 15 \\ &= 5575, \end{aligned}$$

i.e. 5575 pence profit, or £55.75.