Chapter 6

Conditional probability

In this chapter, we look at more complicated notions of probability, and extend the multiplication rule for probability to cater for events that are *not* independent.

6.1 Introduction

So far we have only considered probabilities of single events or of several independent events, like two rolls of a die. However, in reality, many events are related. For example, the probability of it raining in 5 minutes time is dependent on whether or not it is raining now.

We need a mathematical notation to capture how the probability of one event depends on other events taking place. We do this as follows. Consider two events A and B. We write

$$P(A|B)$$

for the probability of A given that B has already happened. We describe $P(A|B)$ as the *conditional probability* of A given B. For example, the probability of it raining in 5 minutes time given that it is raining now would be

$$P(\text{Rain in 5 minutes}|\text{Raining now}).$$

Utility companies need to be able to forecast periods of high demand. They describe their forecasts in terms of probabilities. Gas and electricity suppliers might relate them to air temperature. For example,

$$\begin{aligned} &P(\text{High demand}|\text{air temperature is below normal}) = 0.6\\ &P(\text{High demand}|\text{air temperature is normal}) = 0.2\\ &P(\text{High demand}|\text{air temperature is above normal}) = 0.05. \end{aligned}$$

We can calculate these conditional probabilities using the formula

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)},$$

that is, in terms of the probability of both events occurring, $P(A \text{ and } B)$, and the probability of the event that has already taken place, $P(B)$.

To see how this formula works, let's consider the class of students from Exercises 5.

Student Number	Sex	Height (m)	Weight (kg)	Shoe Size	Student Number	Sex	Height (m)	Weight (kg)	Shoe Size
1	M	1.91	70	11.0	10	M	1.78	76	8.5
2	F	1.73	89	6.5	11	M	1.88	64	9.0
3	M	1.73	73	7.0	12	M	1.88	83	9.0
4	M	1.63	54	8.0	13	M	1.70	55	8.0
5	F	1.73	58	6.5	14	M	1.76	57	8.0
6	M	1.70	60	8.0	15	M	1.78	60	8.0
7	M	1.82	76	10.0	16	F	1.52	45	3.5
8	M	1.67	54	7.5	17	M	1.80	67	7.5
9	F	1.55	47	4.0	18	M	1.92	83	12.0

Suppose we want the probability that a student chosen at random from this class will be female given that the student's shoe size is less than 8. We could simply find the proportion of students with shoe sizes less than 8 who are female. There are 7 students with shoe sizes less than 8 and 4 of these are female. So

$$P(\text{Female}|\text{Shoe size less than 8}) = \frac{4}{7}.$$

This probability can also be calculated using the above formula as follows:

$$\begin{aligned}
P(\text{Shoe size less than 8}) &= \frac{7}{18}, \\
P(\text{Shoe size less than 8 and female}) &= \frac{4}{18}; \qquad \text{thus} \\
P(\text{Female}|\text{Shoe size less than 8}) &= \frac{P(\text{Shoe size less than 8 and female})}{P(\text{Shoe size less than 8})} = \frac{4/18}{7/18} = \frac{4}{7}.
\end{aligned}$$

6.2 Multiplication of probabilities

We saw in Chapter 5 that, if two events A and B are independent, then $P(A \text{ and } B) = P(A) \times P(B)$. Now we know that

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)},$$

we can easily see that

$$P(A \text{ and } B) = P(B) \times P(A|B).$$

Of course it is also true that

$$P(A \text{ and } B) = P(A) \times P(B|A).$$

For example, consider a student chosen at random from the example class. Let F be the event "the student is female" and S be the event "the student's weight is less than 60kg". Then the probability that the student is female and has a weight less than 60kg is

$$\begin{aligned} P(F \text{ and } S) &= P(S) \times P(F|S) = \frac{7}{18} \times \frac{3}{7} = \frac{3}{18} \\ &= P(F) \times P(S|F) = \frac{4}{18} \times \frac{3}{4} = \frac{3}{18}. \end{aligned}$$

Notice that, if M is the event "the student is male", then $P(S|M) = 4/14 = 0.286$ and this is not equal to $P(S|F) = 3/4 = 0.75$. So the probability of the student having a weight less than 60kg depends on the student's sex, that is whether the student is female or male. The events S and F are not independent. Similarly $P(F|S) = 3/7 = 0.429$ while $P(F|L) = 1/11 = 0.091$, where L is the event "the students's weight is not less than 60kg". So, knowing whether or not a student's weight is less than 60kg gives us information about whether the student is likely to be male or female.

Let $\bar{B}$ be the event "not B". So, for example $\bar{F} = M$. Then we say that two events A and B are independent if $P(A|B) = P(A|\bar{B}) = P(A)$. It is easy to show that this is equivalent to $P(B|A) = P(B|\bar{A}) = P(B)$. If A and B are independent then $P(A \text{ and } B) = P(A) \times P(B)$.

Example

For example, consider the following probabilities for customers at a cafe who can choose eiether ice cream or treacle sponge and custard.

	Ice cream	Treacle sponge
Male	0.250	0.150
Female	0.375	0.225

We see that $P(\text{male}) = 0.250 + 0.150 = 0.4$ and $P(\text{female}) = 0.375 + 0.225 = 0.6 = 1 - P(\text{male})$. Now

$$P(\text{Ice cream}|\text{Male}) = \frac{0.250}{0.4} = 0.625$$

and

$$P(\text{Ice cream}|\text{Female}) = \frac{0.375}{0.6} = 0.625,$$

so Ice cream and Male are independent events. In fact, the variables Sex and Dessert–choice are independent in this example. So the probability that a customer is male and chooses ice cream is just $P(\text{Male}) \times P(\text{Ice cream}) = 0.4 \times 0.625 = 0.25$ (the probability of ice cream is just $0.250 + 0.375 = 0.625$).

Example
Another example relates to the age and sex distribution of purchasers of CD singles at an outlet:

	< 30	$30 - 50$	$50+$
Male	0.275	0.125	0.025
Female	0.325	0.175	0.075

From this table, we can calculate

$$\begin{aligned} P(\text{Male}) &= P(\text{Male and } < 30) + P(\text{Male and } 30-50) + P(\text{Male and } 50+) \\ &= 0.275 + 0.125 + 0.025 = 0.425 \end{aligned}$$

and

$$\begin{aligned} P(\text{Female}) &= P(\text{Female and } < 30) + P(\text{Female and } 30-50) + P(\text{Female and } 50+) \\ &= 0.325 + 0.175 + 0.075 = 0.575. \end{aligned}$$

Also, the age distribution of the customers is

$$\begin{aligned} P(<30) &= P(\text{Male and } < 30) + P(\text{Female and } < 30) = 0.275 + 0.325 = 0.6 \\ P(30-50) &= P(\text{Male and } 30-50) + P(\text{Female and } 30-50) = 0.125 + 0.175 = 0.3 \\ P(50+) &= P(\text{Male and } 50+) + P(\text{Female and } 50+) = 0.025 + 0.075 = 0.1. \end{aligned}$$

Using this information we can calculate various probabilities such as:

$$\begin{aligned} P(\text{Male}|30-50) &= \frac{P(\text{Male and } 30-50)}{P(30-50)} = \frac{0.125}{0.3} = 0.4167 \\ P(\text{Female}|30-50) &= 1 - P(\text{Male}|30-50) = 1 - 0.4167 = 0.5833 \end{aligned}$$

and

$$\begin{aligned} P(<30|\text{Male}) &= \frac{P(\text{Male and } < 30)}{P(\text{Male})} = \frac{0.275}{0.425} = 0.6471 \\ P(30-50|\text{Male}) &= \frac{P(\text{Male and } 30-50)}{P(\text{Male})} = \frac{0.125}{0.425} = 0.2941 \\ P(50+|\text{Male}) &= 1 - P(<30|\text{Male}) - P(30-50|\text{Male}) = 1 - 0.6471 - 0.2941 = 0.0588. \end{aligned}$$

6.3 Tree Diagrams

Tree diagrams or probability trees are simple clear ways of presenting probabilistic information. Let us first consider a simple example in which a die is rolled twice. Suppose we are interested in the probability that we score a six on both rolls. This probability can be calculated as

$$\begin{aligned} P(\text{Six and Six}) &= P(\text{Six on 1st throw}) \times P(\text{Six on 2nd throw}|\text{Six on 1st throw}) \\ &= \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{36}. \end{aligned}$$

This example can be represented as a tree diagram in which experiments are represented by circles (called *nodes*) and the outcomes of the experiments as *branches*. The branches are annotated by the probability of the particular outcome.

Here the probability of a six followed by a six is found by tracing the branch corresponding to this outcome through the tree. Note that the ends of the branches of the tree are usually known as *terminal nodes*.

Consider a more complicated example. A machine is used to produce components. Each time it produces a component there is a chance that the component will be defective. When the machine is working correctly the probability that a component is defective is 0.05. Sometimes, though, the machine requires adjustment and, when this is the case, the probability that a component is defective is 0.2. At the time in question there is a probability of 0.1 that the machine requires adjustment. Components produced by the machine are tested and either accepted or rejected. A component which is not defective is accepted with probability 0.97 and (falsely) rejected with probability 0.03. A defective component is (falsely) accepted with probability 0.15 and rejected with probability 0.85.

We can calculate various probabilities. For example:

$$
\begin{aligned}
P(\text{accepted}) &= 0.82935 + 0.00675 + 0.07760 + 0.00300 = 0.9167 \\
P(\text{defective}) &= (0.9 \times 0.05) + (0.1 \times 0.2) = 0.045 + 0.02 = 0.065 \\
P(\text{defective and accepted}) &= 0.00675 + 0.00300 = 0.00975 \\
P(\text{accepted} \mid \text{defective}) &= \frac{0.00975}{0.065} = 0.15 \\
P(\text{defective} \mid \text{accepted}) &= \frac{0.00975}{0.9167} = 0.010636 \\
P(\text{machine OK and accepted}) &= 0.82935 + 0.00675 = 0.8361 \\
P(\text{machine OK} \mid \text{accepted}) &= \frac{0.8361}{0.9167} = 0.9121 \\
P(\text{machine OK and rejected}) &= 0.02565 + 0.03825 = 0.0639 \\
P(\text{rejected}) &= 1 - P(\text{accepted}) = 0.0833 \\
P(\text{machine OK} \mid \text{rejected}) &= \frac{0.0639}{0.0833} = 0.7671
\end{aligned}
$$

6.4 Exercises

1. Do you think the following pairs of events are independent or dependent? Explain.

   (a) E: An individual has a high IQ
   F: An individual is accepted for a University place

   (b) A: A student plays table tennis
   B: A student is good at maths

   (c) E_1: An individual has a large outstanding credit card debt
   E_2: An individual is allowed to extend his bank overdraft

2. A computer retailer conducts a survey of 200 computer purchasers and obtains the following results.

	Age		
	Less than 30	30–44	45 and over
Male	60	20	40
Female	40	30	10

   A customer is selected at random.

   (a) What is the probability that the customer is male and aged 30–44?

   (b) Given that this customer is aged 30–44, what is the probability that they are male?

   (c) Given that this customer is female, what is the probability that they are younger than 45?

3. If Greg goes to the cinema, there is a 60% chance he will then also go to the bar afterwards. However, if he doesn't go to the cinema, this reduces to just 30%. On Friday night, Greg decides to go to the cinema only if his friend Kerry also goes. Greg has no idea about Kerry's intentions this Friday. Let C be the event that Greg goes to the cinema, and B the event that Greg goes to the bar, this Friday. Find

   (a) $P(C)$

   (b) $P(\bar{C})$

   (c) $P(\bar{B}|C)$

   (d) $P(\bar{B}|\bar{C})$

   (e) $P(C \text{ and } B)$

   (f) $P(B)$

4. A company has installed a new computer system and some employees are having difficulty logging on to the system. They have been given training and the problems which arose during training were recorded and their probabilities calculated as follows:

   - An employee has a probability of 0.9 of logging on successfully on the first attempt.
   - If the employee logs in successfully then the employee will also be successful on each later attempt with probability 0.9.
   - If the employee tries to log in and is not successful then the employee loses confidence and the probability of a successful log-in on later occasions drops to 0.5.

   Use a tree diagram to find the following probabilities:

   (a) An employee successfully logs on in each of the first three attempts.

   (b) An employee fails in the first attempt but is successful in the next two attempts.

   (c) An employee logs on successfully only once in three attempts.

   (d) An employee does not manage to log on successfully in three attempts.