Chapter 10

Linear Programming II

10.1 Graphical solutions for two–variable problems

In last week's lecture we discussed how to *formulate* a linear programming problem; this week, we consider how to *solve* such problems. Linear programming problems that involve only two decision variables, x and y, may be solved graphically, that is by drawing lines associated with the constraints, identifying a region of possible solutions and then locating a point in this region that has a particular property (i.e. a point which maximises profit or minimises cost). To solve linear programming problems using graphical methods you need to be able to use the following basic mathematical ideas.

10.1.1 Sets of points defined by a linear inequality

Any equation of the form $ax+by=c$, where a, b and c are numbers, is called a **linear equation**. For example, $3x+4y=12$ is a linear equation. In the x,y plane this is the equation of a straight line, and this line may be drawn by identifying any two points on it (since it only takes two points to draw a straight line).

For example, for the equation $3x+4y=12$, we can identify two points which lie on the straight line by considering what happens when $x=0$ and what happens when $y=0$.

- When $x=0$, we have

$$\begin{aligned} 3\times 0+4y &= 12 \qquad \text{i.e.} \\ 0+4y &= 12 \qquad \text{i.e.} \\ 4y &= 12 \qquad \text{i.e.} \\ y &= 3. \end{aligned}$$

  So one point on the line $3x+4y=12$ is at $x=0, y=3$, or $(0,3)$ (remember, we go "along the corridor and up the stairs" when plotting points given by coordinates, so for this point we'd go "0 along, and 3 up").

- When $y = 0$, we have

$$\begin{aligned} 3x + 4 \times 0 &= 12 \quad \text{i.e.} \\ 3x + 0 &= 12 \quad \text{i.e.} \\ 3x &= 12 \quad \text{i.e.} \\ x &= 4. \end{aligned}$$

So another point on the line $3x + 4y = 12$ is at $x = 4, y = 0$, or $(4, 0)$.

Since we have the coordinates of two points which lie on the line, and the line is a straight line, we can plot the line with this equation! (figure 10.1)

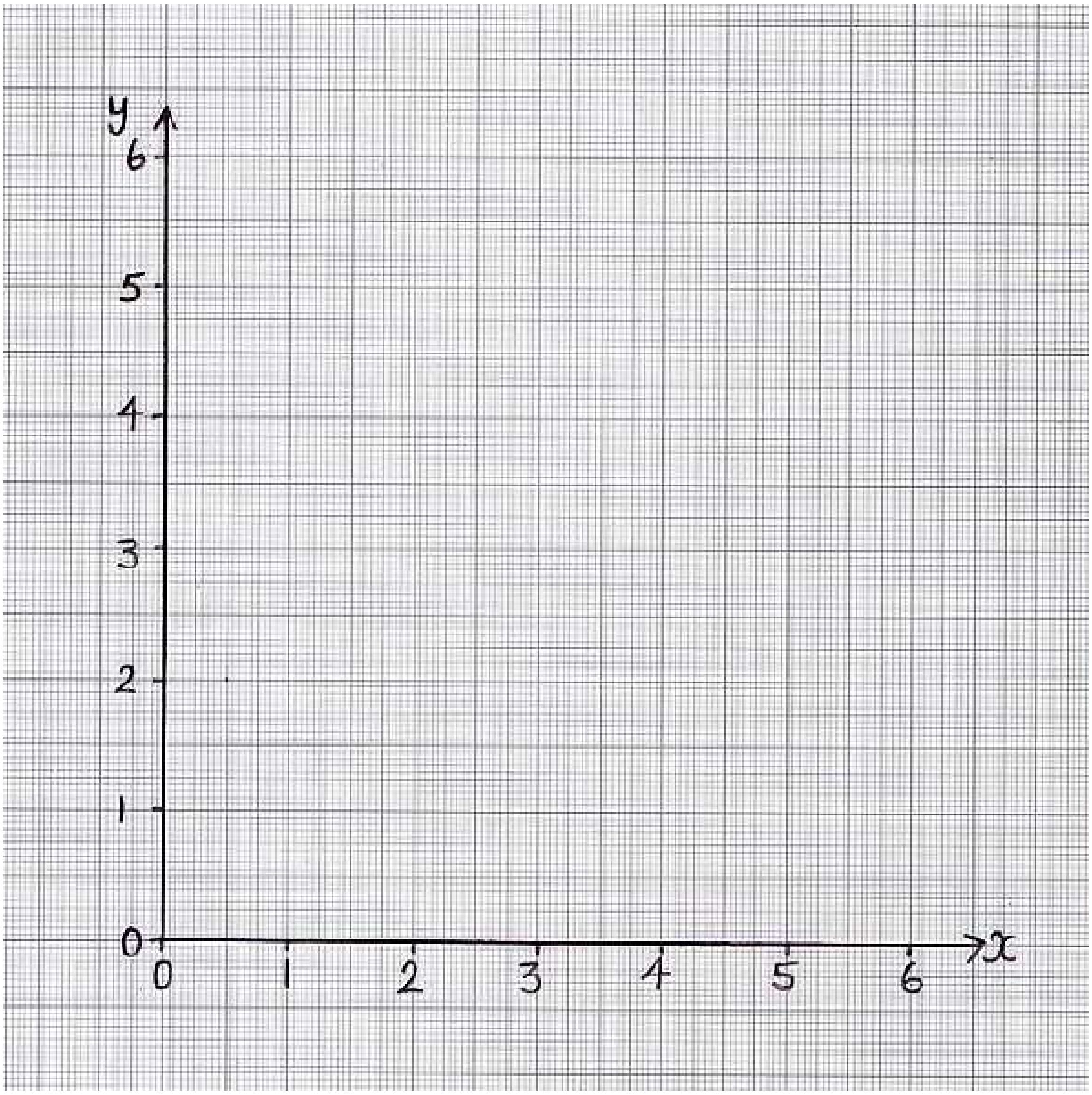

Figure 10.1: Plot of the line $3x + 4y = 12$

Any straight line divides the xy–plane into two half–planes. If the equation of the line is $ax + by = c$ then on one side of the line you have $ax + by < c$ and on the other side of the line you have $ax + by > c$.

Example
Show on a diagram the region for which $3x + 4y \leq 12$.

The first step is to draw the line $3x + 4y = 12$ (as in figure 10.1). We are interested in the region which is *less than or equal to 12*; we can find out where abouts this is by seeing if the origin lies in the required region or not. Now at the origin, $x = 0$ and $y = 0$, so we have $3x + 4y = 3 \times 0 + 4 \times 0 = 0$, which is less than 12. Thus, the origin *does* lie in the required region, and the half–plane required contains the origin. So all points on the line or below it satisfy the inequality. We call this the **admissible set** defined by the inequality. We show this on the graph by shading out points which do not lie in the required region, or shading out the **inadmissible set** of points (see figure 10.2).

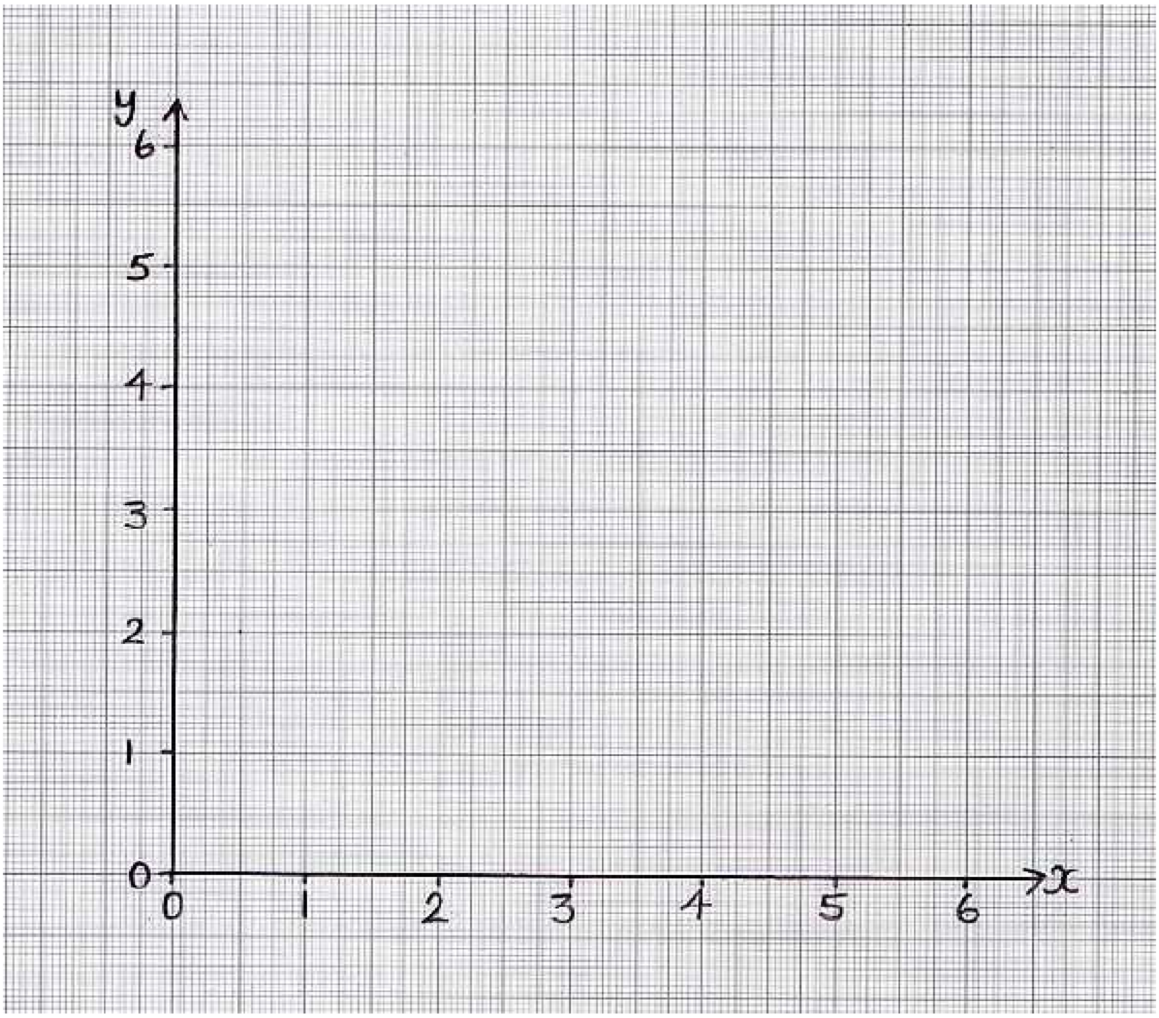

Figure 10.2: Indicating the region for which $3x + 4y \leq 12$

10.1.2 Sets of points defined by a collection of inequalities

Each inequality in a linear programming problem will produce an admissible set. To find a solution to the problem you need to find the set of points which satisfies all the inequalities *simultaneously*. This is obtained graphically by drawing a diagram like the one in figure 10.2 but showing *all* the inequalities. **The required region is then the one which does not contain any shading marks**.

Example
Indicate on a diagram the region for which

$$\begin{aligned} 3x + 4y &\leq 12, \\ 3x + 2y &\leq 9, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

We've already looked at how to show the admissible set for the first inequality $3x + 4y \leq 12$; the same procedure can be used for the inequality $3x + 2y \leq 9$. Firstly, we need to plot the line $3x + 2y = 9$. Again, it's easiest to consider what happens when $x = 0$ and $y = 0$. For example,

- When $x = 0$, we have

$$\begin{aligned} 3 \times 0 + 2y &= 9 \qquad \text{i.e.} \\ 0 + 2y &= 9 \qquad \text{i.e.} \\ 2y &= 9 \qquad \text{so} \\ y &= 4.5. \end{aligned}$$

- Similarly, when $y = 0$, we have

$$\begin{aligned} 3x + 2 \times 0 &= 9 \qquad \text{i.e.} \\ 3x + 0 &= 9 \qquad \text{i.e.} \\ 3x &= 9 \qquad \text{so} \\ x &= 3. \end{aligned}$$

So the points $x = 0, y = 4.5$ and $x = 3, y = 0$, i.e. $(0, 4.5)$ and $(3, 0)$ lie on the line with equation $3x + 2y = 9$. Figure 10.3 shows a plot of the line with this equation. The plot also shows the set of admissible points for our inequality; remember, we want the set of points such that $3x + 2y \leq 9$, so the region we require lies on, or underneath, the line drawn. The inadmissible set has been shaded out.

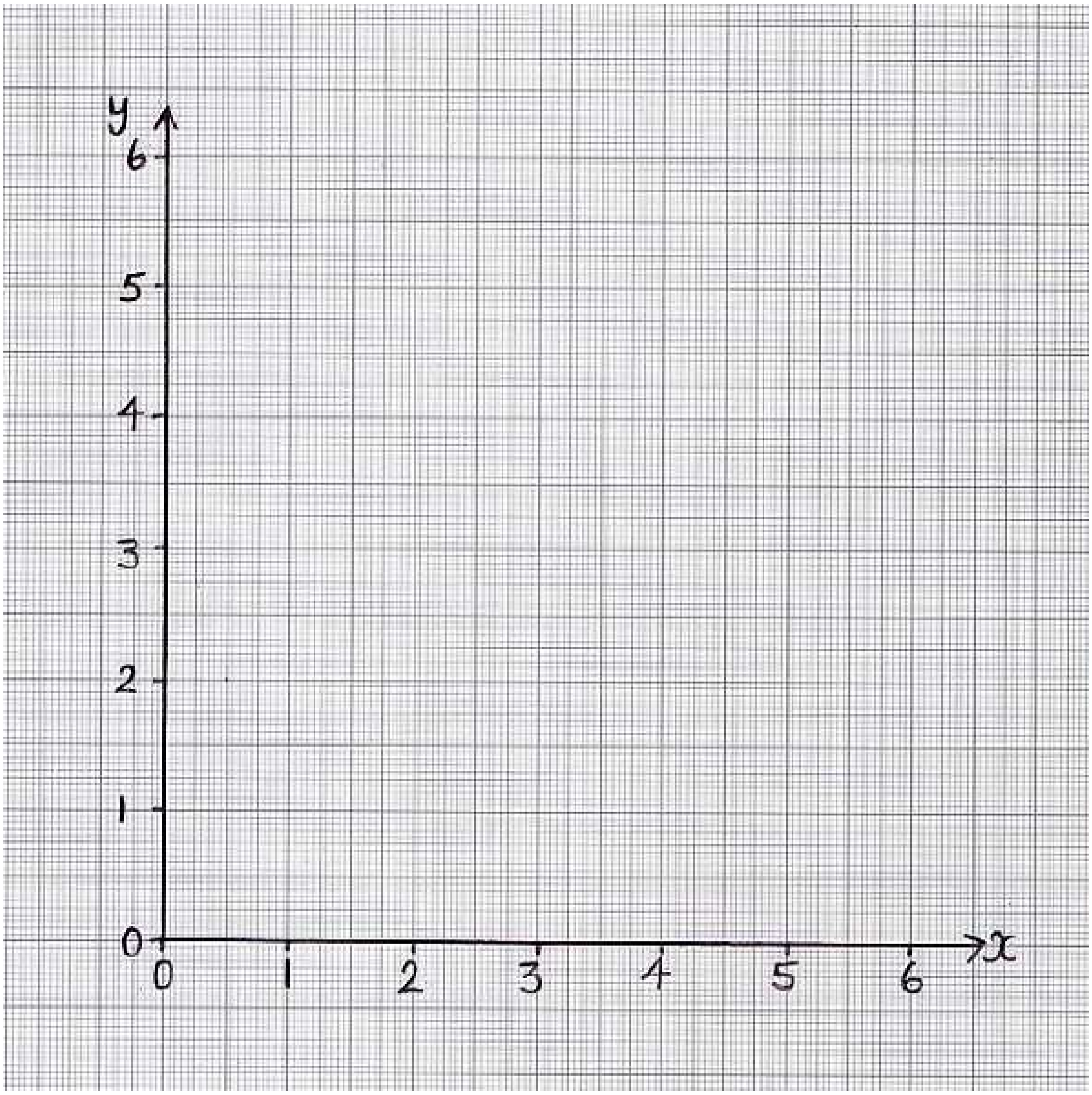

Figure 10.3: Plot of the line $3x + 2y = 9$; also shown is the admissible region for the inequality $3x + 2y \leq 9$

The inequality $x \geq 0$ gives: and the inequality $y \geq 0$ gives:

Combining these with the inequality $3x + 4y \leq 12$ considered previously, we get

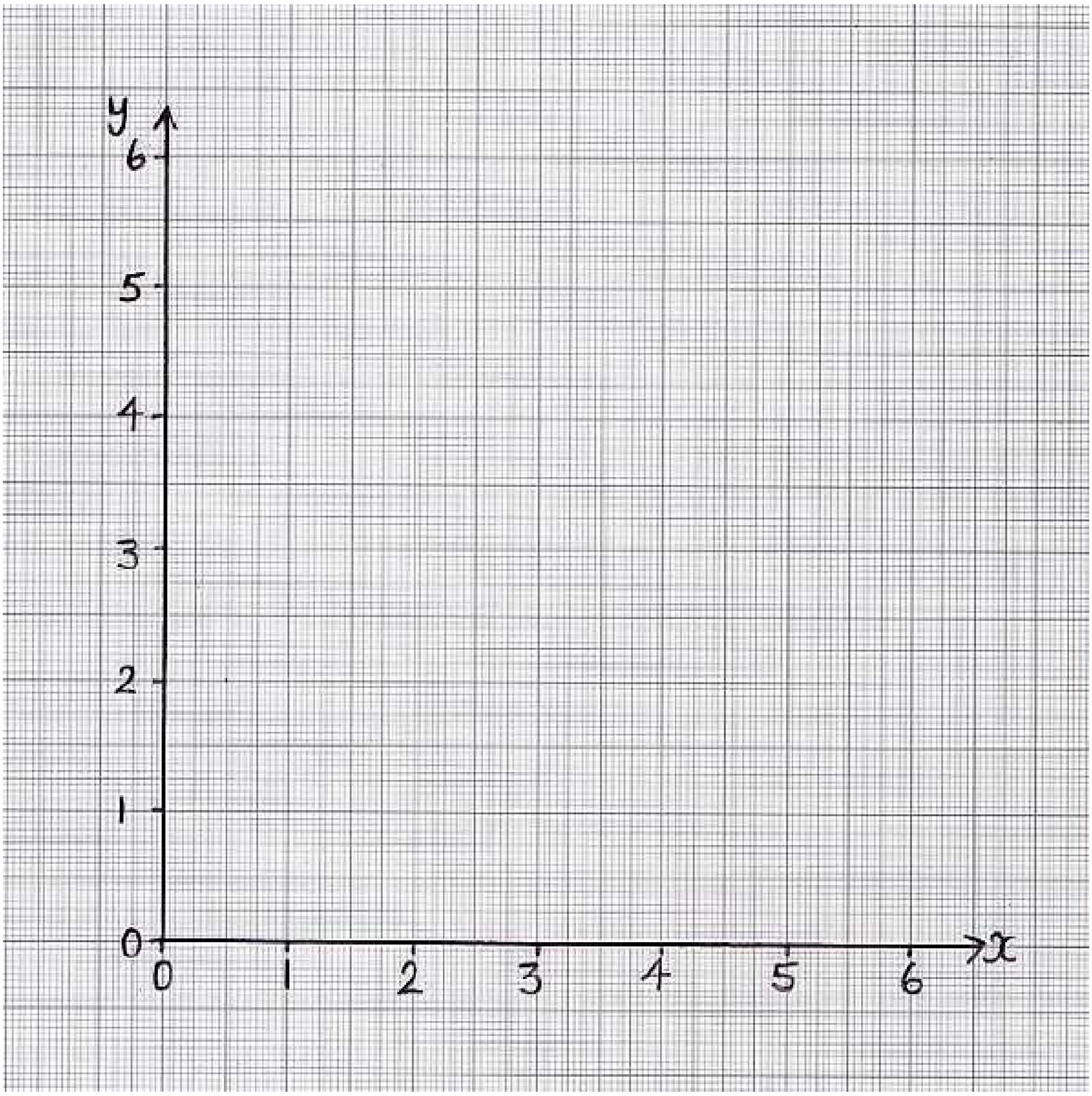

Figure 10.4: Diagram showing the admissible region satisfying the inequalities $3x + 4y \leq 12$, $3x + 2y \leq 9$, $x \geq 0$ and $y \geq 0$

We now use these techniques to obtain feasible solutions to the linear programming problems discussed last week, and look at how to find the *optimal* solution for each of these problems.

Recall the three scenarios:

1. a chair manufacturer,
2. a book publisher, and
3. a haulage company.

10.2 Solving linear programming problems

10.2.1 Example (A chair manufacturer)

Recall the problem posed in example 9.2.1:

A manufacturer makes two kinds of chairs, A and B, each of which has to be processed in two departments, I and II. Chair A has to be processed in department I for 3 hours and in department II for 2 hours. Chair B has to be processed in department I for 3 hours and in department II for 4 hours.

The time available in department I in any given month is 120 hours, and the time available in department II, in the same month, is 150 hours.

Chair A has a selling price of £10 and chair B has a selling price of £12.

The manufacturer wishes to maximise his income. How many of each chair should be made in order to achieve this objective? You may assume that all chairs made can be sold.

Last week we considered how to *formulate* this situation as a linear programming problem; using the graphical techniques outlined in section 10.1, we will now consider how to *solve* this problem.

Recall that things are made much easier if we summarise the information given in a table; we can then read down the columns to obtain the inequalities for the linear programming problem. The table below summarises the information given in this particular problem.

Type of chair	Time in dept. I (hours)	Time in dept. II (hours)	Selling price (£)
A	3	2	10
B	3	4	12
Total time available	120	150	

Recall the three steps involved in formulating a linear programming problem:

1. Identify the **decision variables**
2. Identify the **constraints**
3. State the **objective function**

In this example, the decision variables were identified as

$$\begin{aligned} x &= \text{number of type A chairs made} \qquad \text{and} \\ y &= \text{number of type B chairs made.} \end{aligned}$$

The constraints were

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

We then identified that the objective was to maximise the income, which we called Z, where

$$Z = 10x + 12y.$$

So, in summary, the linear programming problem is:

Maximise $Z = 10x + 12y$ subject to the following constraints:

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

To find the **feasible region** for this problem (i.e. the region which satisfies all of our inequalities), we proceed by indicating, on a diagram, the region for which all of the inequalities hold (as we did in the example in section 10.1.2).

The first inequality is $3x + 3y \leq 120$. To show this on a diagram, we first need to plot the line $3x + 3y = 120$.

- When $x = 0$, we have

$$\begin{aligned} 3 \times 0 + 3y &= 120 \qquad \text{i.e.} \\ 3y &= 120 \qquad \text{i.e.} \\ y &= 40. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 3x + 3 \times 0 &= 120 \qquad \text{i.e.} \\ 3x &= 120 \qquad \text{i.e.} \\ x &= 40. \end{aligned}$$

These points are plotted on figure 10.5, and the line with equation $3x + 3y = 120$ drawn. Since we want $3x + 3y \leq 120$, our region of interest lies on or below the line, and so we shade out the space above the line.

Now consider the second inequality $2x + 4y \leq 150$. Again, to show this on a diagram, we first need to plot the line $2x + 4y = 150$.

- When $x = 0$, we have

$$\begin{aligned} 2 \times 0 + 4y &= 150 \qquad \text{i.e.} \\ 4y &= 150 \qquad \text{i.e.} \\ y &= 37.5. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 2x + 4 \times 0 &= 150 \qquad \text{i.e.} \\ 2x &= 150 \qquad \text{i.e.} \\ x &= 75. \end{aligned}$$

Again, these points are plotted on figure 10.5, and the line with equation $2x + 4y = 150$ drawn. Since we want $2x + 4y \leq 150$, our region of interest lies on or below the line, and so we shade out the space above the line.

On figure 10.5 we also shade out the inadmissible regions for the two non–negativity constraints.

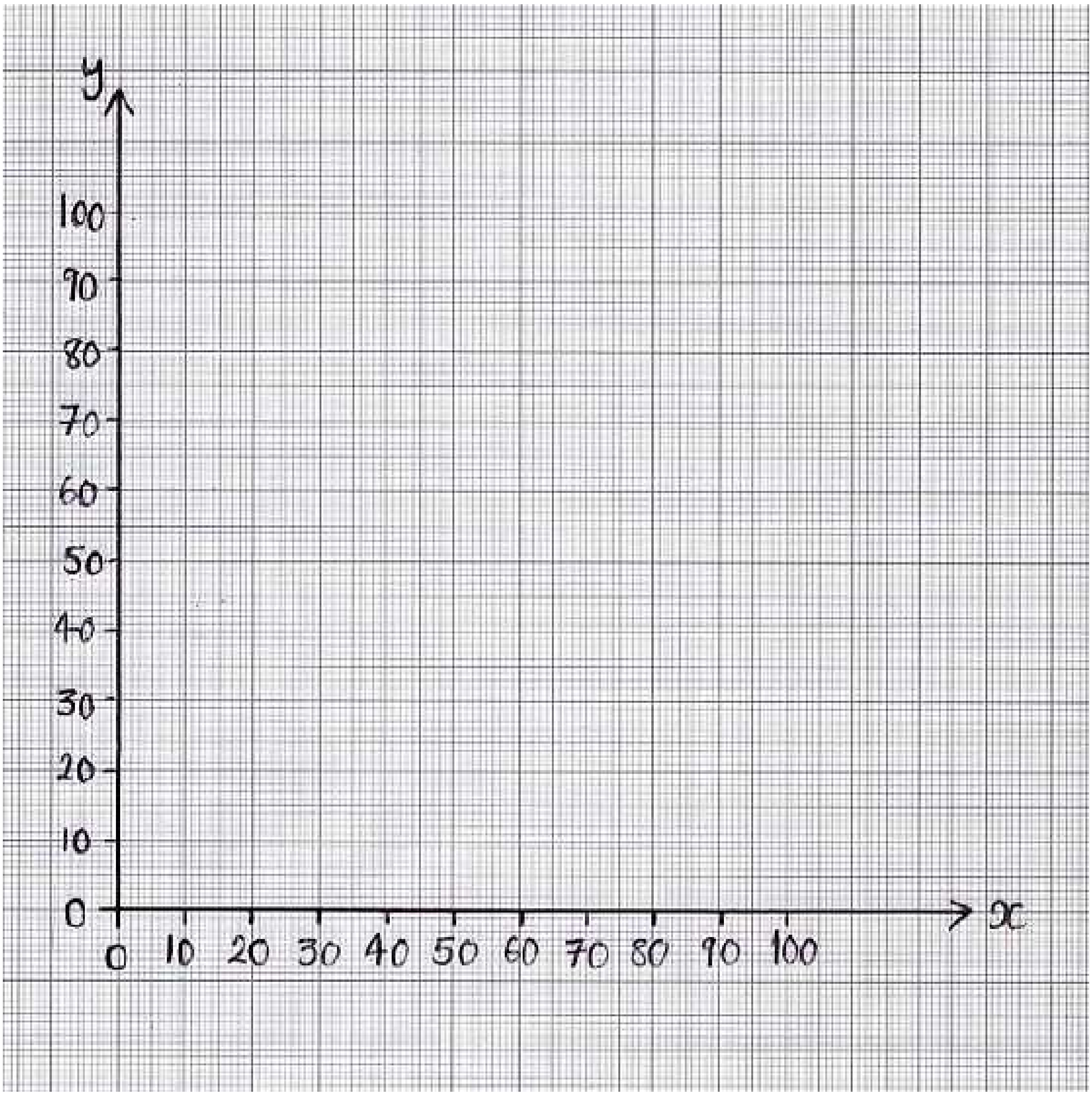

Figure 10.5: Feasible region and objective function for the chair manufacturing problem

The unshaded region in figure 10.5 shows the feasible region associated with our set of inequalities. What we must do now is find the point in that region which meets our objective – i.e. the point in that region which maximises income. One way of doing this is to also plot the objective function. Now our objective function is

$$Z = 10x + 12y,$$

where Z is our income. When Z takes different values we get a family of parallel straight lines. We need to choose a starting value for Z in order to be able to plot the objective function. It's often a good idea to try a value which is a multiple of both the coefficients of x and y. The coefficient of x is 10 and the coefficient of y is 12, so we could try a starting value of $Z = 120$. Thus, the objective function is now

$$10x + 12y = 120.$$

We can plot this line in the same way as before – i.e. consider what happens when x and y are zero.

- When $x = 0$, we have

$$\begin{aligned} 10 \times 0 + 12y &= 120 \quad \text{i.e.} \\ 12y &= 120 \quad \text{i.e.} \\ y &= 10. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 10x + 12 \times 0 &= 120 \quad \text{i.e.} \\ 10x &= 120 \quad \text{i.e.} \\ x &= 12. \end{aligned}$$

This line is also plotted on figure 10.5. Notice that this line does not give the optimal income; the origin represents zero income, and we want to move as far away from this as possible. The largest value of Z (income) will occur at the point in the feasible region that is *furthest* from the origin, but still parallel to the objective line. You can find this point by sliding a ruler over the feasible region so that it is always parallel to the objective function drawn. This will enable you to identify the point that is furthest from the origin. In fact, you should notice that the point in the feasible region furthest from the origin (and parallel to the objective function) is the intersection of the two lines with equations $3x + 3y = 120$ and $2x + 4y = 150$.

All points in the feasible region satisfy our inequalities, but only one point maximises income. Once this point has been identified, we can simply "read off" the x and y values. Doing so give $x = 5$ and $y = 35$, and so, in order to maximise income, we should make 5 type A chairs and 35 type B chairs. This will give an income of

$$\begin{aligned} Z &= 10x + 12y \quad \text{i.e.} \\ Z &= 10 \times 5 + 12 \times 35 \quad \text{i.e.} \\ Z &= 50 + 420 \\ &= 470, \end{aligned}$$

i.e. £470.

10.2.2 Example (A book publisher)

Recall the problem considered in example 9.2.2:

A book publisher is planning to produce a book in two different bindings: paperback and library. Each book goes through a sewing process and a gluing process. The sewing process is available for 7 hours per day and the gluing process for 15 hours per day. The profits are 25p on a paperback edition and 60p on a library edition. How many books in each binding should be manufactured to maximise profits? (assume that the publisher sells as many of each type of book as is produced.)

The table below summarises the information given in the above paragraph:

	Sewing (mins)	Gluing	Profit (P)
Paperback	2	4	25
Library	3	10	60
Total time	420 (in minutes!)	900 (in minutes!)	

The linear programming problem is summarised below:

Maximise $P = 25x + 60y$ subject to the constraints

$$\begin{aligned} 2x + 3y &\leq 420, \\ 4x + 10y &\leq 900, \\ x &\geq 0 \quad \text{and} \\ y &\geq 0. \end{aligned}$$

As before, we solve this problem by first of all drawing the feasible region satisfying all of our inequalities. For the inequality $2x + 3y \leq 420$, we draw the line $2x + 3y = 420$ and then shade out the area *above* the line (since the required region is less than, or equal to, 420). So:

- When $x = 0$, we have

$$\begin{aligned} 2 \times 0 + 3y &= 420 \quad \text{i.e.} \\ 3y &= 420 \quad \text{i.e.} \\ y &= 140. \end{aligned}$$

- Similarly, when $y = 0$, we have

$$\begin{aligned} 2x + 3 \times 0 &= 420 \quad \text{i.e.} \\ 2x &= 420 \quad \text{i.e.} \\ x &= 210. \end{aligned}$$

We consider the inequality $4x + 10y \leq 900$ in a similar way; we draw the line with equation $4x + 10y = 900$ and then shade out the region above this line.

- When $x = 0$, we have

$$\begin{aligned} 4 \times 0 + 10y &= 900 \quad \text{i.e.} \\ 10y &= 900 \quad \text{i.e.} \\ y &= 90. \end{aligned}$$

- Similarly, when $y = 0$, we have

$$\begin{aligned} 4x + 10 \times 0 &= 900 \quad \text{i.e.} \\ 4x &= 900 \quad \text{i.e.} \\ x &= 225. \end{aligned}$$

Both these lines are plotted on figure 10.6, and the inadmissible regions shaded out. Also shaded out are the inadmissible regions for the two non–negativity constraints.

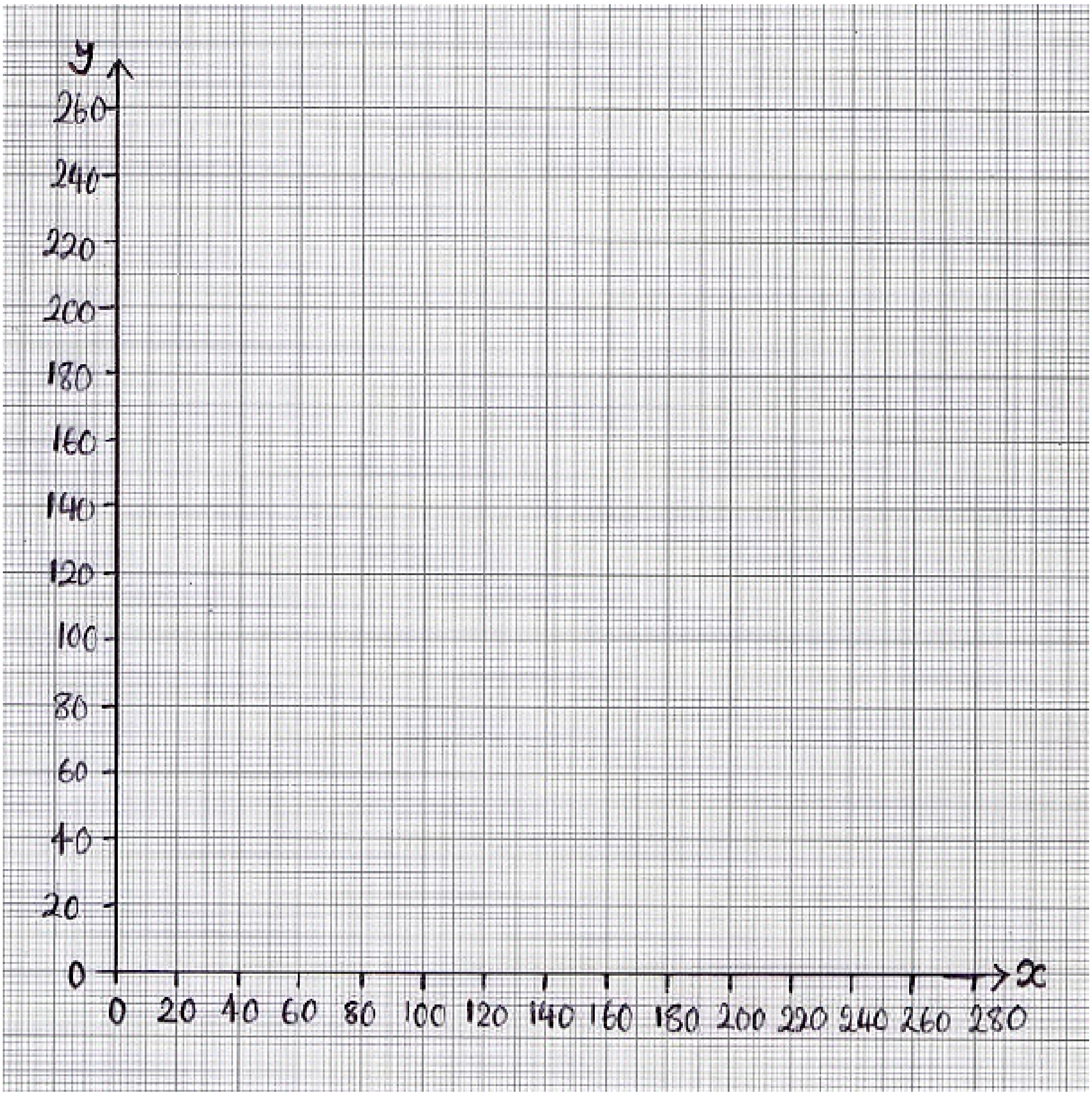

Figure 10.6: Feasible region and objective function for the book publisher's problem

We now need to plot the objective function. The objective is to maximise profit, P, where

$$P = 25x + 60y.$$

Of course, we don't yet know what P is, so we choose a starting value. As before, it's convenient to choose a value which is a multiple of both the x and y coefficients. The x coefficient is 25 and the y coefficient is 60, so we choose $25 \times 60 = 1500$. Thus, our objective function is

$$25x + 60y = 1500.$$

To plot this line, we consider what happens when x and y are zero.

- When $x = 0$, we have

$$\begin{aligned} 25 \times 0 + 60y &= 1500 \quad \text{i.e.} \\ 60y &= 1500 \quad \text{i.e.} \\ y &= 25. \end{aligned}$$

- Similarly, when $y = 0$ we have

$$\begin{aligned} 25x + 60 \times 0 &= 1500 \quad \text{i.e.} \\ 25x &= 1500 \quad \text{i.e.} \\ x &= 60. \end{aligned}$$

This line is plotted on figure 10.6. As before, we need to move this line as far away from the origin as possible in order to maximise profits, but keep the line inside the feasible region. You can see that the point in the feasible region which is furthest away from the origin, but parallel to the objective line, is the intersection between the lines with equations $2x + 3y = 420$ and $4x + 10y = 900$. As before, we can "read off" our solutions from the graph. Doing so, gives $x = 187.5$ and $y = 15$. Thus, the book publisher should make 187.5 paperback bindings and 15 library bindings in order to maximise profit. Since it doesn't really make sense to make 187.5 paperback bindings, we round down to 187. Thus, the publisher should make 187 paperback bindings and 15 library bindings in order to maximise profit. This will give

$$\begin{aligned} P &= 25x + 60y \\ &= 25 \times 187 + 60 \times 15 \\ &= 5575, \end{aligned}$$

i.e. 5575 pence profit, or £55.75.

10.2.3 Example (A haulage company)

This one's left for you to try!

10.3 Exercises

1. Indicate on a diagram the region for which

$$\begin{aligned} 4x+3y &\leq 12, \\ 2x+5y &\leq 10, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

2. Solve the linear programming problem formulated in question 1 of exercises 9. [*Hint: Try a starting value for the profit of £600*]

3. Kuddly Pals Co. Ltd make two types of giant soft toy: bears and cats. The quantity of material needed and the time taken to make each type of toy is given in the table below.

Toy	Material (m^2)	Time (minutes)
Bear	5	12
Cat	8	8

Each day the company can process up to 2000m^2 of material and there are 48 worker hours available to assemble the toys.

The profit made on each bear is £1.50 and on each cat is £1.75. Kuddly Pals Co. Ltd wishes to maximise its daily profit.

(i) Formulate the company's situation as a linear programming problem.

(ii) Draw a suitable diagram to enable the problem to be solved graphically, indicating the feasible region and the direction of the objective line.

(iii) Use your diagram to find the company's maximum profit, £P.