Lecture 9

LINEAR PROGRAMMING (I)

Introduction

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Two aims often considered in business are:

- **maximising profit**, and
- **minimising cost**.

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- **requirements**
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as algebraic equations. They then developed methods for obtaining the **optimal solution** to the problem posed.

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For our set of algebraic equations to reflect the **requirements**, **constraints** and **objectives** of a real–life situation, you can imagine how complex they would be!

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For example, we might express **profit** as a linear combination of two other variables:

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This is a linear **equation**.

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In today's lecture, we will consider how to **formulate** real–life situations as linear programming problems.

Next week, we will discuss how to **solve** such problems.

Formulating the problem

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2. Identify the **constraints**
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3. Determine the **objective function**
   This is the quantity to be be *optimised*, usually **profit** or **costs**.

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- A **haulage company**.

Example 1: A chair manufacturer

A manufacturer makes two kinds of chairs – **A** and **B**. Each type of chair has to be processed in two departments – $\mathbf{I}$ and $\mathbf{II}$.

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Chair **A** spends 3 hours in department $\mathbf{I}$ and 2 hours in department $\mathbf{II}$. Chair **B** spends 3 hours in department $\mathbf{I}$ and 4 hours in department $\mathbf{II}$.

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Chair **A** has a selling price of £10 and chair **B** of £12.

The manufacturer wishes to maximise his income.

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How many of each type of chair should be made?

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Chair	Time in dept. I	Time in dept. II	Selling price
A	3	2	10
B	3	4	12
Time limits	120	150	

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1. What are the **decision variables**? (i.e. which quantities do you need to know in order to solve the problem?)
2. What are the **constraints**?
3. What is the **objective**?

Step 1: Decision variables

Read through the question and identify the things you'd like to know. You can usually do this by going straight to the last sentence of the question:

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"How many of each chair should be made..."

Thus, we'd like to know

- the number of type **A** chairs to make, and
- the number of type **B** chairs to make.

These are our decision variables, and are usually denoted with lower case letters. Thus, our decision variables are

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$$x \quad = \quad \text{number of type } \mathbf{A} \text{ chairs made} \qquad \text{and}$$

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$$\begin{aligned} x &= \text{number of type } \textbf{A} \text{ chairs made} \qquad \text{and} \\ y &= \text{number of type } \textbf{B} \text{ chairs made.} \end{aligned}$$

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Since:	the production of 1 type **A** chair uses 3 hours,
then:	the production of x type **A** chairs takes $3x$ hours.

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Since:	the production of 1 type **A** chair uses 3 hours,
then:	the production of x type **A** chairs takes $3x$ hours.
Similarly:	the production of 1 type **B** chair uses 3 hours,
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Since:	the production of 1 type **A** chair uses 2 hours,
then:	the production of x type **A** chairs takes $2x$ hours.
Similarly:	the production of 1 type **B** chair uses 4 hours,
so:	the production of y type **B** chairs takes $4y$ hours.

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Since only 150 hours are available in department $\mathbf{II}$, a second constraint is

$$\begin{aligned} (2x + 4y) \text{ hours} &\leq 150 \text{ hours}, \qquad \text{or just} \\ (2x + 4y) &\leq 150. \end{aligned}$$

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These are called the **non–negativity constraints**.

Step 3: Objective function

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If we make x type **A** chairs, then we get $£10 \times x = £10x$, since each type **A** chair sells for $£10$.

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If we make x type **A** chairs, then we get £$10 \times x =$ £$10x$, since each type **A** chair sells for £10.

Similarly, if we make y type **B** chairs, then we get £$12 \times y =$ £$12y$, since each type **B** chair sells for £12.

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where Z is the objective function.

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Example 2: A book publisher

A book publisher is planning to produce a book in two different bindings: paperback and library. Each book goes through a sewing process and a gluing process. The table below gives the time required, in minutes, for each process and for each of the bindings:

	Sewing (mins)	Gluing (mins)
Paperback	2	4
Library	3	10

The sewing process is available for 7 hours per day and the gluing process for 15 hours per day.

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The profits are 25p on a paperback edition and 60p on a library edition.

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The profits are 25p on a paperback edition and 60p on a library edition.

How many books in each binding should be manufactured to maximise profits?

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	Sewing (mins)	Gluing	Profit (P)
Paperback	2	4	25
Library	3	10	60
Total time	420 (in minutes!)	900 (in minutes!)	

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