

Lecture 8

TIME SERIES AND FORECASTING

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The first step in the analysis of time series is to plot the data against time in a **time series plot**.

Describing time series

The lecture notes give the commands for producing time series plots in Minitab. Note that the axes should be labelled appropriately and that the plot should have a title.

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- Does the plot show **both** trend **and** seasonality?
- Are there any **outliers**?
- Does the plot look **stationary**?

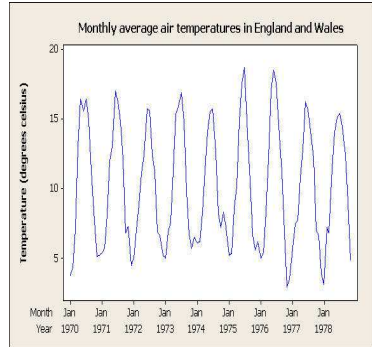
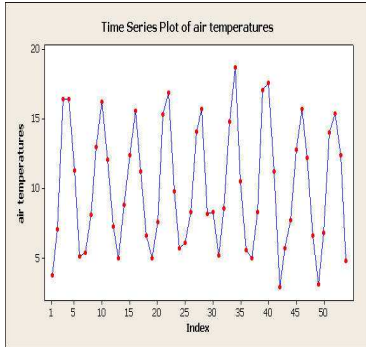


Figure: Left: Time series plots of monthly average air temperatures in England and Wales (right-hand-side plot shows edited time axis and inserted title)

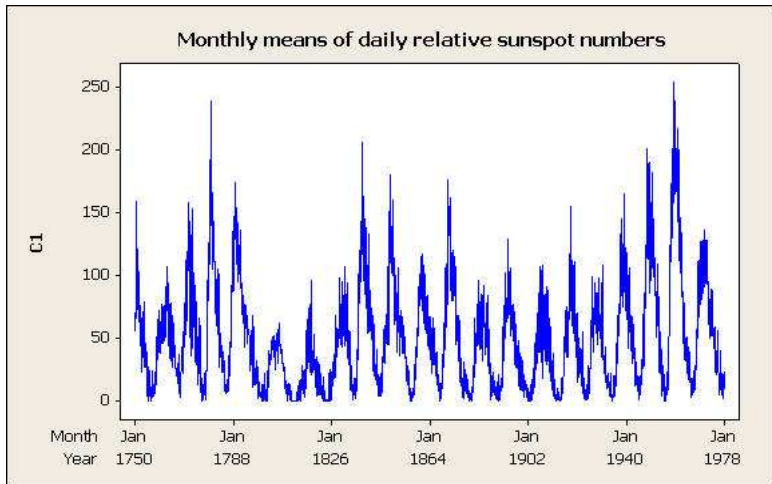


Figure: Time series plot of the monthly means of daily relative sunspot numbers

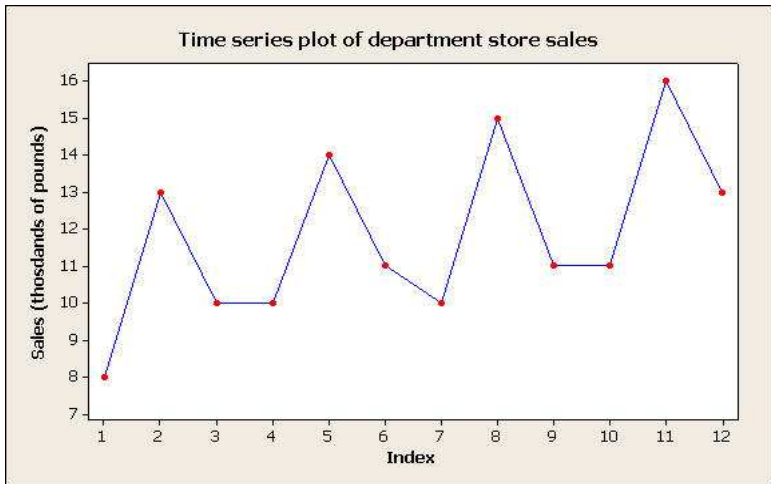


Figure: Time series plot of four-monthly sales for a department store

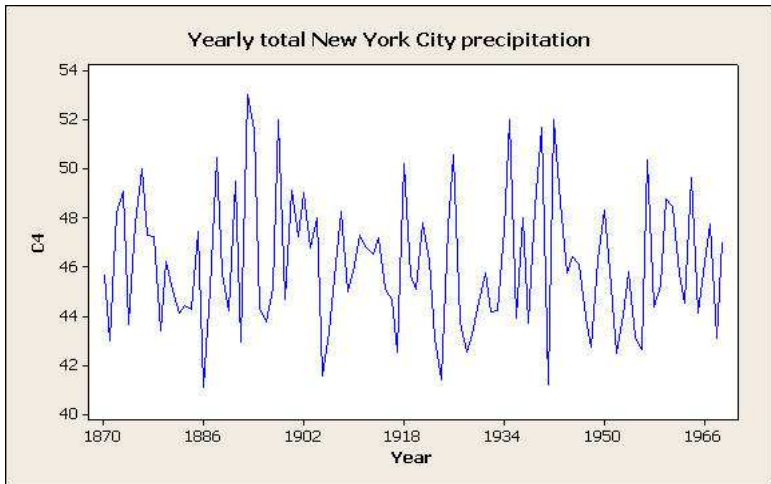


Figure: Time series plot yearly total precipitation in New York

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	Jan–Apr	May–Aug	Sep–Dec
1994	8	13	10
1995	10	14	11
1996	10	15	11
1997	11	16	13

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- To calculate moving averages, you first have to decide on the **cycle length**. This is the number of “seasons” you have, and is usually pretty obvious.
- In our example there are **3** seasons (Jan–Apr, May–Aug and Sep–Dec).

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We proceed in this way to calculate all the moving averages. Note that there's no moving average for the first time point or the last time point.

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The calculated moving averages are shown in the table below. A plot of these against time, along with the original data, is shown in the figure on the next slide.

	Moving averages		
	Jan-Apr	May-Aug	Sep-Dec
1994	*	10.33	11.00
1995	11.33	11.67	11.67
1996	12.00	12.00	12.33
1997	12.67	13.33	*

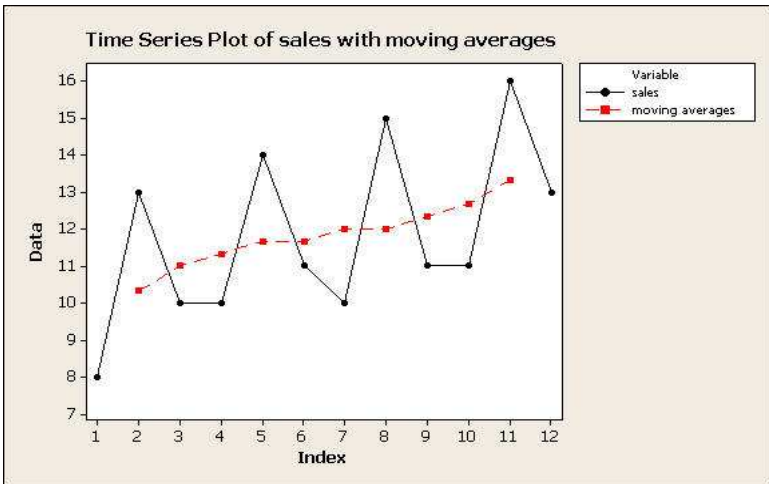


Figure: Time series plot of sales, with overlaid moving averages

Using regression to estimate the trend

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$$\begin{aligned}\hat{\alpha} &= \bar{y} - \hat{\beta}\bar{x} & \text{and} \\ \hat{\beta} &= \frac{S_{XY}}{S_{XX}}\end{aligned}$$

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So if we wanted to predict sales in the period Jan–Apr 1998, we would substitute $T = 13$ into the regression equation above, since the last observed time point was for Sep–Dec 1997 and was the 12th time point.

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So if we wanted to predict sales in the period Jan–Apr 1998, we would substitute $T = 13$ into the regression equation above, since the last observed time point was for Sep–Dec 1997 and was the 12th time point.

But, before we do this, we need to estimate α and β !

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Remember, the easiest way to calculate these quantities is to draw up a table!

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t	y (moving averages)	ty	t^2
2	10.33	20.66	4
3	11.00	33.00	9
4	11.33	45.32	16
5	11.67	58.35	25
6	11.67	70.02	36
7	12.00	84.00	49
8	12.00	96.00	64
9	12.33	110.97	81
10	12.67	126.7	100
11	13.33	146.63	121
65	118.33	791.65	505

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$$\begin{aligned}S_{TT} &= \left(\sum t^2 \right) - n\bar{t}^2 \\ &= 505 - 10 \times 6.5 \times 6.5 \\ &= \mathbf{82.5}.\end{aligned}$$

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$$\begin{aligned}\hat{\alpha} &= \bar{y} - \hat{\beta}\bar{t} \\ &= 11.83 - 0.275 \times 6.5 \\ &= \mathbf{10.043}.\end{aligned}$$

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So, the regression equation for our trend is

$$\mathbf{Y = 10.043 + 0.275T + \epsilon.}$$

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We now look at how to **estimate these cycles**, known as **seasonal effects**.

Calculating seasonal effects

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Similarly, for Sep–Dec 1994, we have

$$10.00 - 11.00 = -1.$$

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	seasonal deviations		
	Jan–Apr	May–Aug	Sep–Dec
1994	*	2.67	–1
1995	–1.33	2.33	–0.67
1996	–2	3	–1.33
1997	–1.67	2.67	*
means	–1.67	2.6675	–1

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$$\begin{aligned}\frac{2.67 - 1 - 1.33 + \dots - 1}{10} &= \frac{2.67}{10} \\ &= 0.267.\end{aligned}$$

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$$\begin{aligned}\frac{-1.937 + 2.4005 - 1.267}{3} &= \frac{-0.8035}{3} \\ &= -0.26783.\end{aligned}$$

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Just be careful with double negatives! Now the seasonal effects *do* sum to give zero!

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To predict average sales in Jan–Apr 1998, we would substitute $T = 13$ into the above equation, since this would be our 13th observation. Doing so, gives

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$$\begin{aligned} Y &= 10.043 + 0.275 \times 13 \\ &= 10.043 + 3.575 \\ &= 13.618. \end{aligned}$$

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$$13.618 + (-1.66917) = 11.949,$$

i.e. £11,949, or just under £12,000.