Chapter 2

Optimisation using calculus

What we'll cover...

- The gradient of a **curve**? Ideas of **rates of change**
- The method of **differentiation**: first principles
- Differentiation: **general results** for polynomials
- Finding the derivative of non–linear functions from **real–life** scenarios
- **Optimisation** using the differential calculus, and the role of the second derivative
- Chapters 3 and 4 in the **course text**

2. Optimisation using calculus

An important topic in many disciplines is the study of how quickly quantities **change over time**.

For example, in order to estimate the future demand for a commodity, we need information about **rates of change**.

As we shall see in this chapter, we can also use such information to solve optimisation problems: for example, we might be interested in the number of items that need to be produced in order to **maximise profit** or **minimise loss**.

We have already considered rates of change in Chapter 1.

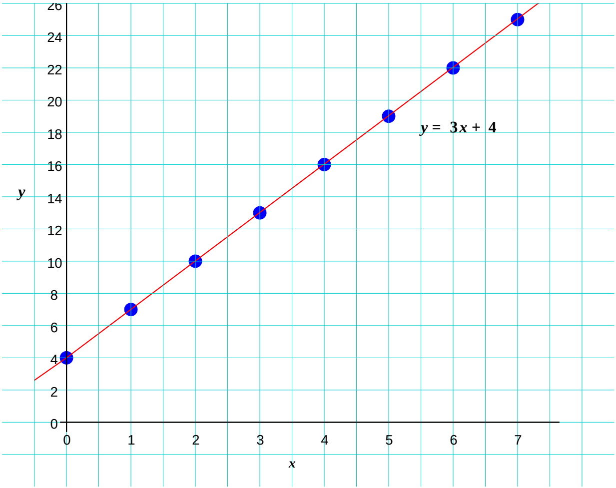
In the equation of the straight line

$$y = mx + c,$$

the gradient m represents the rate of change of y with respect to x, and we thought about how to calculate m using the formula

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x}.$$

Example



2. Optimisation using calculus

In Chapter 1 we also considered some **non–linear functions**, focussing primarily on quadratic functions.

We will begin Chapter 2 with an examination of how to find the **gradient of a non–linear function**, before thinking about how this can be used to aid the process of **optimisation**.

2.1 Differentiation

The concept used to describe the rate of change of a function (linear or non–linear) is the **derivative**, which is the central concept in mathematical analysis.

In this section we will define the derivative of a function, as well as present some of the important rules for calculating it.

Isaac Newton (1642–1727) and **Gottfried Leibniz** (1646–1716) discovered most of these general rules independently of each other.

This initiated the **differential calculus**, which has been the foundation for the development of modern science.

It has also been of central importance to the theoretical development of modern economics.

2.1.1 The gradient of a curve

A curve does not have a constant gradient – its direction is **continuously changing**, and so its gradient will continuously change too.

So we look at the gradient of the curve at a **particular point on the curve**, rather than calculate the gradient of an entire line as with a linear function.

2.1.1 The gradient of a curve

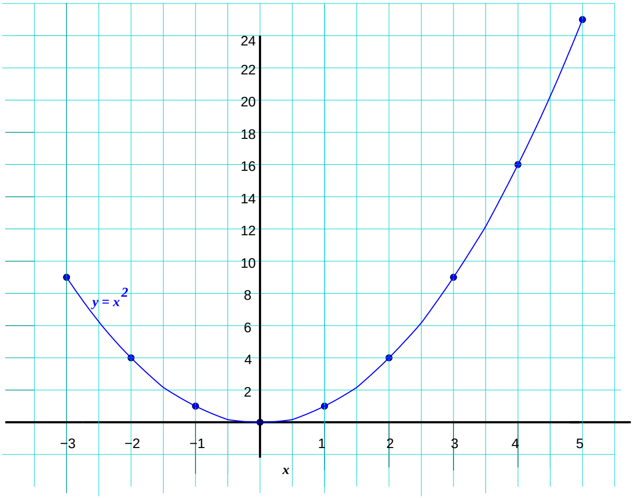
Look at the graph of $y = x^2$ shown below.

Let's suppose we are interested in the gradient of this curve at the point $x = 3$.

Informally, we can say a couple of things about the gradient at this point:

- The gradient will be **positive** – as you look at the graph from left to right at this point, the curve goes uphill
- The graph is **symmetric** about the point $x = 0$: the gradient at $x = -3$ will be the same as that at $x = 3$ (just negative)
- As we move from $x = 3$ to $x = 0$, the gradient gets **less and less steep**, and "flattens out" completely at $x = 0$

2.1.1 The gradient of a curve



2.1.1 The gradient of a curve

We know how to find the gradient of a straight line from Chapter 1, so we can **approximate** the gradient of the curve at $x = 3$ by drawing a chord on the curve.

For example, draw a chord on the curve between the points $x = 3$ and $x = 5$, and find the gradient of this chord *[the co-ordinates will be (3,9) and (5,25)]*:

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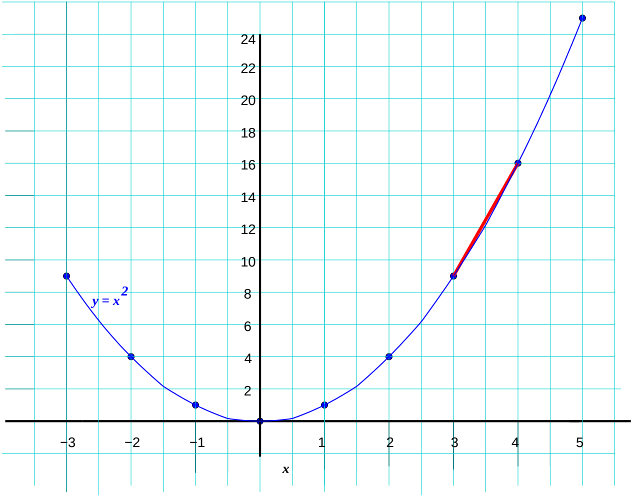
For example, draw a chord on the curve between the points $x = 3$ and $x = 5$, and find the gradient of this chord *[the co-ordinates will be (3,9) and (5,25)]*:

$$\text{Gradient} = \frac{25 - 9}{5 - 3} = \frac{16}{2} = 8.$$

Now decrease the length of the chord and bring it **closer to the curve** – for example, draw the chord between the points $x = 3$ and $x = 4$ – and find the gradient *[The co-ordinates will be (3,9) and (4,16)]*:

2.1.1 The gradient of a curve

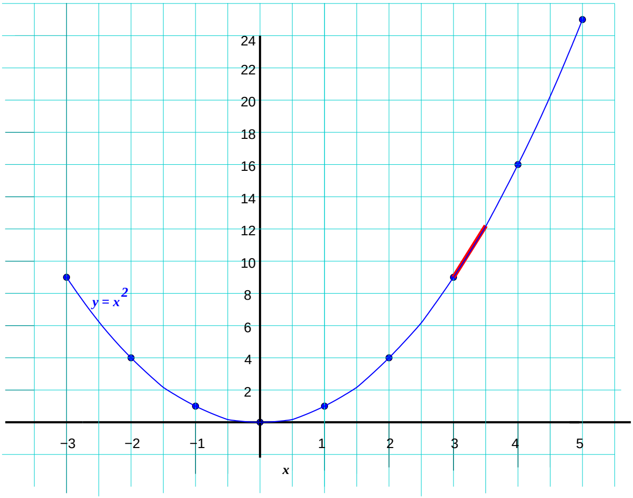


Now decrease the length of the chord and bring it **closer to the curve** – for example, draw the chord between the points $x = 3$ and $x = 4$ – and find the gradient *[The co-ordinates will be (3,9) and (4,16)]*:

$$\text{Gradient} = \frac{16 - 9}{4 - 3} = \frac{7}{1} = 7.$$

Now decrease the length of the chord again and bring it **even closer to the curve** – draw the chord between the points $x = 3$ and $x = 3.5$ – and find the gradient *[The co-ordinates will be (3,9) and (3.5,12.25)]*:

2.1.1 The gradient of a curve



Now decrease the length of the chord again and bring it **even closer to the curve** – draw the chord between the points $x = 3$ and $x = 3.5$ – and find the gradient *[The co-ordinates will be (3,9) and (3.5,12.25)]*:

$$\text{Gradient} = \frac{12.25 - 9}{3.5 - 3} = \frac{3.25}{0.5} = 6.5.$$

2.1.1 The gradient of a curve

Point of interest	Point 2	Gradient
$(3,9)$	$(5,25)$	$\frac{25-9}{5-3}=\mathbf{8}$
$(3,9)$	$(4,16)$	$\frac{16-9}{4-3}=\mathbf{7}$
$(3,9)$	$(3.5,12.25)$	$\frac{12.25-9}{3.5-3}=\mathbf{6.5}$
$(3,9)$	$(3.25,10.5625)$	$\frac{10.5625-9}{3.25-3}=\mathbf{6.25}$
$(3,9)$	$(3.1,9.61)$	$\frac{9.61-9}{3.1-3}=\mathbf{6.1}$
$(3,9)$	$(3.01,9.0601)$	$\frac{9.0601-9}{3.01-3}=\mathbf{6.01}$

So it looks like at $x = 3$ the gradient converges to 6.

We can **generalise** this approach **algebraically**.

Suppose the point of interest has co–ordinates (x, x^2) and point 2 has an x co–ordinate close to that of the point of interest, say $x + \delta$. Then the y co–ordinate of point 2 is

$$y = (x + \delta)^2.$$

Thus, the gradient is given by

$$\begin{aligned}
\text{Gradient} &= \frac{\text{Change in } y}{\text{Change in } x} \\
&= \frac{(x+\delta)^2 - x^2}{(x+\delta) - x} \\
&= \frac{(x+\delta)(x+\delta) - x^2}{x+\delta-x} \\
&= \frac{x^2 + 2\delta x + \delta^2 - x^2}{\delta} \\
&= \frac{2\delta x + \delta^2}{\delta} = 2x + \delta.
\end{aligned}$$

Now if we let δ get smaller and smaller, i.e. "tend to zero", we are left with just

$$\text{Gradient} = 2x.$$

So the gradient, or **derivative**, of the curve $y = x^2$ is $2x$.

We usually denote this quantity as $\frac{dy}{dx}$, pronounced “dee y by dee x”, and so when

$$\begin{aligned} y &= x^2 \\ \frac{dy}{dx} &= 2x. \end{aligned}$$

We can now use this derivative to find the gradient of the curve at any point x.

For example, when $x = 3$, the gradient is

$$\frac{dy}{dx} = 2x = 2 \times 3 = 6,$$

exactly the answer we found from first principles in the table on page 41!

What is the gradient of the curve $y = x^2$ at the points $x = 4$, $x = 1$, $x = 0$ and $x = -4$? Do your answers make sense in relation to the graph of this function?

Solution

$$x = 4 \quad : \quad \frac{dy}{dx} = 2 \times 4 = 8$$

$$x = 1 \quad : \quad \frac{dy}{dx} = 2 \times 1 = 2$$

$$x = 0 \quad : \quad \frac{dy}{dx} = 2 \times 0 = 0$$

$$x = -1 \quad : \quad \frac{dy}{dx} = 2 \times -1 = -2.$$

Looking at the graph, these values make perfect sense: as we approach zero, we can see the curve becomes less and less steep, until at the origin the gradient is zero.

Also, the curve is symmetric about $x = 0$ and so the gradient at $x = \pm 1$ will be the same, just negated.

2.1.2 Some general results for polynomials

In the previous section we found that, when

$$y = x^2,$$

$$\frac{dy}{dx} = 2x.$$

Notice that the power of x has been "brought down", in front of the x, and the power itself has reduced by one ($2x$ is actually $2x^1$).

2.1.2 Some general results for polynomials

In general, when

$$y = x^n,$$

$$\frac{dy}{dx} = nx^{n-1}.$$

More generally still, if

$$y = kx^n, \qquad \text{then}$$

$$\frac{dy}{dx} = nkx^{n-1}.$$

2.1.2 Examples

Find $\dfrac{dy}{dx}$ for each of the following:

1. $y = x^3$
2. $y = x^5$
3. $y = 2x^3$
4. $y = 4x^3 - 7x^2 + 2x$
5. $y = 5x^2 + 2$
6. $y = \sqrt{x} + 1$
7. $y = \dfrac{1}{x}$

Solution to examples

1. We have $y = x^3$. Bringing the power down, and then reducing the power by one, gives

$$\frac{dy}{dx} = 3x^2.$$

2. We have $y = x^5$. Bringing the power down, and then reducing the power, gives

$$\frac{dy}{dx} = 5x^4.$$

Solution to examples

3. We have $y = 2x^3$. Bringing the power down, and then reducing the power by one, gives

$$\frac{dy}{dx} = \mathbf{3} \times 2x^2 = 6x^2.$$

4. We have $y = 4x^3 - 7x^2 + 2x^1$. Bringing the powers of x down, and then reducing the powers, gives

$$\begin{aligned}\frac{dy}{dx} &= \mathbf{3} \times 4x^2 - \mathbf{2} \times 7x^1 + \mathbf{1} \times 2x^0 \\ &= 12x^2 - 14x + 2.\end{aligned}$$

5. We have $y = 5x^2 + 2$, i.e. $y = 5x^2 + 2x^0$. Thus

$$\frac{dy}{dx} = \mathbf{2} \times 5x^1 + \mathbf{0} \times 2x^{-1} = 10x,$$

so constants just differentiate to zero!

6. We have $y = \sqrt{x} + 1 = x^{0.5} + 1$. Thus

$$
\begin{aligned}
\frac{dy}{dx} &= 0.5x^{-0.5} \\
&= \frac{1}{2}x^{-\frac{1}{2}} \\
&= \frac{1}{2} \times \frac{1}{x^{\frac{1}{2}}} \\
&= \frac{1}{2} \times \frac{1}{\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

7. We have $y = \dfrac{1}{x} = x^{-1}$. Thus

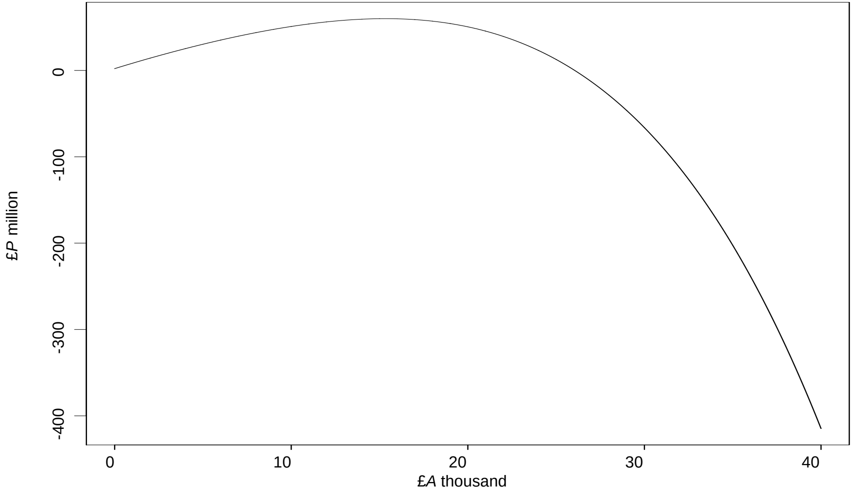
$$\begin{aligned}\frac{dy}{dx} &= -1x^{-2} \\ &= -\frac{1}{x^2}.\end{aligned}$$

Example 2.1

The managing director of a company is interested in modelling the relationship between her company's annual profits (£P million) and their annual advertising budget (£A thousand).

The graph below shows how the managing director believes P changes with A.

Example 2.1



Example 2.1

(a) You work as part of a team of business analysts for this company. A colleague proposes the following three profit functions:

* $P = 80 - 4.3A$
* $P = 2.1 + 13.5A + 0.5A^2$
* $P = 2.1 + 6.3A - 0.13A^2 + 0.0008A^3 - 0.0002A^4$

Which of these do you think would be most appropriate in this situation? Why?

Solution to Example 2.1(a)

We clearly do not have a linear function, and so this rules out the first.

Our graph shows that the function will not be symmetric, and so we cannot have a quadratic.

Thus the quartic must be the most suitable function.

Example 2.1

(b) For the profit function you have chosen in part (a), find $\frac{dP}{dA}$.

Solution to Example 2.1(b)

We have $P = 2.1 + 6.3A^1 - 0.13A^2 + 0.0008A^3 - 0.0002A^4$.

Thus

$$\frac{dP}{dA} = \mathbf{1} \times 6.3A^0 - \mathbf{2} \times 0.13A^1 + \mathbf{3} \times 0.0008A^2 - \mathbf{4} \times 0.0002A^3$$

$$= 6.3 - 0.26A + 0.0024A^2 - 0.0008A^3.$$

(c) Use your answer to part (b) to find the gradient of your profit function when the company spends (i) £10,000; (ii) £15,250; (iii) £20,000 and (iv) 40,000 on advertising each year. Do these values match up with what you see in the graph?

Solution to Example 2.1(c)

$$A = 10 \quad : \quad \frac{dP}{dA} = 6.3 - 0.26 \times 10 + 0.0024 \times 10^2 - 0.0008 \times 10^3 = \mathbf{3.14}$$

$$A = 15.25 \quad : \quad \frac{dP}{dA} = \ldots = \mathbf{0.056}$$

$$A = 20 \quad : \quad \frac{dP}{dA} = \ldots = \mathbf{-4.34}$$

$$A = 40 \quad : \quad \frac{dP}{dA} = \ldots = \mathbf{-51.46}$$

Example 2.1

(d) What will be the value of $\frac{dP}{dA}$ when the company optimises their advertising budget (i.e. maximises profit)?

Solution to Example 2.1(d)

From the graph, we can see that the company will maximise their profit at the peak of the curve.

This is known as a **turning point** in the graph.

- **Before** this point, the gradient ($\frac{dP}{dA}$) is positive
- **After** this point, the gradient ($\frac{dP}{dA}$) is negative
- **At** this point $\frac{dP}{dA} = 0$

Example 2.2

A small firm employs five machine operators.

For a particular contract, the firm believes it will produce $Q = \frac{1}{2}\sqrt{M}$ units of a commodity, where M machine operators are used.

The cost, per operator, is €40 and the price obtained per unit is €160.

(a) Formulate a non–linear function for the firm's profit, π, in terms of M only.

Solution to Example 2.2(a)

$$\begin{aligned}
\text{Profit} &= \text{Total income} - \text{Total costs} \\
\pi &= 160Q - 40M \\
&= 160\left(\frac{1}{2}\sqrt{M}\right) - 40M \\
&= 80\sqrt{M} - 40M.
\end{aligned}$$

Example 2.2

(b) Find $\frac{d\pi}{dM}$, and solve $\frac{d\pi}{dM} = 0$ for M. Comment.

Solution to Example 2.2(b)

We have $\pi = 80\sqrt{M} - 40M = 80M^{0.5} - 40M$. Thus

$$\begin{aligned}\frac{d\pi}{dM} &= 0.5 \times 80M^{-0.5} - 40\\ &= 40 \times \frac{1}{M^{0.5}} - 40\\ &= \frac{40}{\sqrt{M}} - 40.\end{aligned}$$

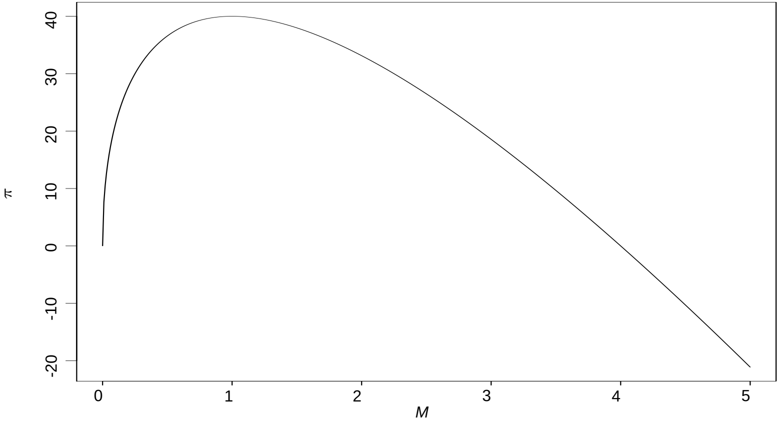
When $\frac{d\pi}{dM} = 0$ we get

$$\begin{aligned}\frac{40}{\sqrt{M}} &= 40\\ 40 &= 40\sqrt{M}\\ 1 &= \sqrt{M},\end{aligned}$$

and so $M = 1$. This means that the turning point in our graph will occur when we use 1 machine operator.

Example 2.2

(c) Shown below is a plot of π against M. Can you see how this corresponds to your answer to part (b)?



Solution to Example 2.2(c)

It is clear from the graph that the turning point in this function occurs at $M = 1$ – our calculations show that this is the point at which the gradient is zero.

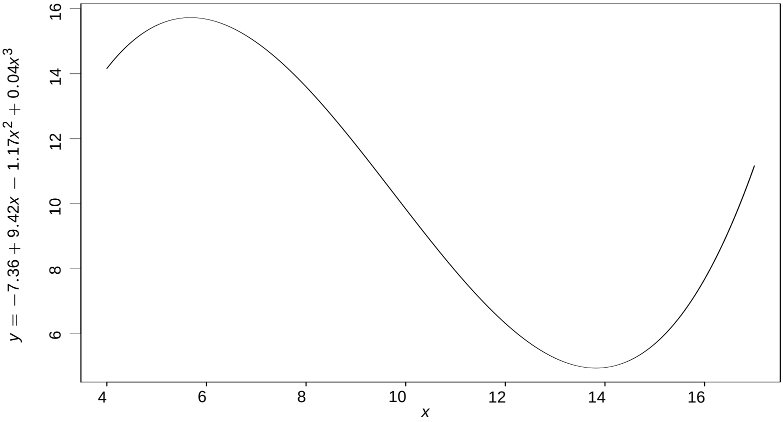
Suppose a company believes there is a non–linear relationship between it's monthly advertising budget (£x thousand) and their monthly profit (£y thousand).

In particular, their analyst believes that the following cubic function explains how y varies with x:

$$y = -7.36 + 9.42x - 1.17x^2 + 0.04x^3.$$

The company's monthly advertising budget cannot exceed £17,000; their current contractual arrangements for commercial TV and radio advertising means they must always spend at least £4,000 on advertising every month.

A graph of this cubic, between the values of $x = 4$ and $x = 17$, is shown overleaf.


$y = -7.36 + 9.42x - 1.17x^2 + 0.04x^3$
6
8
10
12
14
16
4
6
8
10
12
14
16
x

Some things to note from the graph:

- There are two **turning points** on the graph – one at about $x = 6$ and the other at about $x = 14$;
- One is known as a **local maximum**, the other is a **local minimum**;
- In this example, one of these points corresponds to maximum profit, the other corresponds to minimum profit;
- At both turning points, the gradient of the curve – and hence $\frac{dy}{dx}$ – is **zero**.

To find the exact x co–ordinate of each turning point, we can equate the derivative to zero and solve for x.

For example, we know that

$$\begin{aligned}\frac{dy}{dx} &= 9.42 - 2 \times 1.17x + 3 \times 0.04x^2 \\ &= 9.42 - 2.34x + 0.12x^2.\end{aligned}$$

At each of the turning points shown in the graph, we know that the gradient is zero. Thus, we know that

$$9.42 - 2.34x + 0.12x^2 = 0,$$

and we know how to solve such a quadratic equation from the material in Chapter 1!

2.2 Optimisation using differentiation

Using the quadratic formula, we know that

$$x = \frac{-b \pm \sqrt{D}}{2a},$$

where D is the discriminant and is equal to $b^2 - 4ac$; a and b are the coefficients of the x^2 and x terms, respectively, and c is the constant.

In this example, $a = 0.12$, $b = -2.34$ and $c = 9.42$, giving

$$D = (-2.34)^2 - (4 \times 0.12 \times 9.42) = 0.954,$$

and so

$$x = \frac{2.34 \pm \sqrt{0.954}}{0.24},$$

giving

$$x = \frac{2.34 + \sqrt{0.954}}{0.24} = 13.81971, \qquad \text{or}$$

$$x = \frac{2.34 - \sqrt{0.954}}{0.24} = 5.680295.$$

These are the **precise** values of *x* at which the curve has zero gradient – i.e. the advertising expenditure that will minimise/maximise profit – and this can be seen in the graph above.

So, spending £5,680 per month on advertising seems to be the optimal strategy, which will give monthly profits of

$$y = -7.36 + 9.42(5.680) - 1.17(5.680^2) + 0.04(5.680^3) = £15,729.$$

Spending £13,820 per month on advertising seems to give the worst outcome, with monthly profits of just

$$y = -7.36 + 9.42(13.820) - 1.17(13.820^2) + 0.04(13.820^3) = £4,944.$$

It is obvious from our graph which of the turning points gives the maximum profit and which gives the minimum profit.

However, if we didn't have a graph of the function, the **second derivative test** can help us here.

The second derivative of a function, sometimes written as

$$\frac{d^2y}{dx^2},$$

("dee 2 y by dee x squared") is just the derivative of the derivative.

If x represents the x co–ordinate of our turning point, i.e. $\frac{dy}{dx}(x) = 0$, then:

- If $\frac{d^2y}{dx^2}(x) < 0$, our turning point is a **maximum**;
- If $\frac{d^2y}{dx^2}(x) > 0$, our turning point is a **minimum**.

In this example,

$$\frac{dy}{dx} = 9.42 - 2.34x + 0.12x^2.$$

Thus,

$$\frac{d^2y}{dx^2} = -2.34 + 0.24x.$$

At the first turning point, $x = 5.680295$, giving

$$\frac{d^2y}{dx^2} = -2.34 + 0.24 \times 5.680295 = -0.9767292,$$

which is negative, and so here we have a **maximum turning point**.

At the second turning point $x = 13.81971$, giving

$$\frac{d^2y}{dx^2} = -2.34 + 0.24 \times 13.81971 = 0.9767304,$$

which is positive, and so here we have a **minimum turning point**.

To summarise:

- There are two turning points: one at $x = 5.680295$ and one at $x = 13.81971$
- At $x = 5.680295$, the second derivative is negative and so we have a local maximum
- At $x = 13.81971$ the second derivative is positive and so we have a local minimum
- The local maximum/minimum correspond to maximum/minimum profit, and so spending £5,680 pounds on advertising is the optimal strategy, giving profits of just under £16,000.
- The worst strategy is to spend £13,820 on advertising, which will give a profit of just under £5,000.

Example 2.3

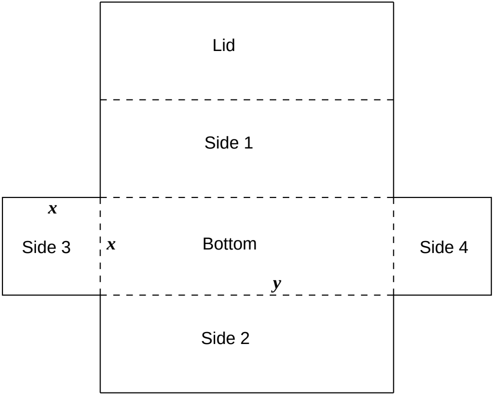
Choctastic! are a chocolate manufacturer.

One of their gift boxes has 27 chocolates in it, requiring a volume of 1350 cubic centimetres.

The company would like to work out the optimal dimensions of the required box to minimise the amount of packaging used (and hence minimise production costs).

Example 2.3

The diagram below shows a "net" of the box that will be used; x and y are both in centimetres.



Example 2.3: Solution to part (a)

(a) Write down an expression for the volume of the box, in terms of x and y.

Solution

The volume is given by

$$\begin{aligned} \text{length} \times \text{width} \times \text{height} &= 1350 \\ y \times x \times x &= 1350 \\ x^2 y &= 1350. \end{aligned}$$

Example 2.3: Solution to part (b) (1/2)

(b) Formulate an expression for the surface area S of the box, in terms of x only.

Solution

Looking at the net, there are four rectangles with area xy and two squares with area x^2. So the total surface area is

$$S = 4xy + 2x^2.$$

Example 2.3: Solution to part (b) (2/2)

Now from (a), we know that

$$y = \frac{1350}{x^2}.$$

Substituting this into the equation for S gives

$$\begin{aligned} S &= 4x\left(\frac{1350}{x^2}\right) + 2x^2 \\ &= \frac{4x \times 1350}{x^2} + 2x^2 \\ &= \frac{5400}{x} + 2x^2 = 5400x^{-1} + 2x^2. \end{aligned}$$

Example 2.3

(c) Find $\frac{dS}{dx}$, and hence show that the optimal strategy is to use a cuboid box for the chocolates. What are the dimensions of this cuboid?

Example 2.3: Solution to part (c) (1/3)

We have

$$S = 5400x^{-1} + 2x^2.$$

Thus,

$$\frac{dS}{dx} = -5400x^{-2} + 4x.$$

At turning points the gradient is zero. Thus

$$\begin{aligned} -\frac{5400}{x^2} + 4x &= 0 \\ 4x &= \frac{5400}{x^2} \end{aligned}$$

Example 2.3: Solution to part (c) (2/3)

So

$$\begin{aligned} 4x^3 &= 5400 \\ x^3 &= 1350 \\ x &= \sqrt[3]{1350} = 11.052 \text{ cm.} \end{aligned}$$

Example 2.3: Solution to part (c) (3/3)

When $x = 11.052$,

$$y = \frac{1350}{x^2} = \frac{1350}{11.052^2} = 11.052 \text{ cm};$$

with reference to the net, we see this will give a cuboid shape for the box!

Example 2.3

(d) Use the second derivative test to show that you have minimised the amount of packaging used.

We know that

$$\frac{dS}{dx} = -5400x^{-2} + 4x.$$

Thus

$$\frac{d^2S}{dx^2} = 10800x^{-3} + 4.$$

At our turning point, $x = 11.052$, giving

$$\frac{d^2S}{dx^2} = \frac{10800}{11.052^3} + 4 = 12,$$

which is positive – this confirms that our turning point is a **minimum**.

As you will see in some of your other courses, most relationships in Economics, Accounting and Finance involve **more than two variables**.

For example, the demand for a good depends not only on its own price but also on the price of substitutable or complementary goods, incomes of consumers, advertising expenditure and so on.

Likewise, the output from a production process depends on a variety of inputs, including land, capital and labour.

To analyse general economic behaviour we must extend the concept of differentiation to several variables.

Examples

1. If $f(x, y) = xy + 2y$, evaluate

(a) $f(3, 4)$;

(b) $f(4, 3)$.

$$\begin{aligned} f(3,4) &= 3 \times 4 + 2 \times 4 = 20 \\ f(4,3) &= 4 \times 3 + 2 \times 3 = 18 \end{aligned}$$

2. If $g(x_1, x_2, x_3) = x_1^2 + x_2 - 3x_3$, evaluate

(a) $g(5, 6, 10)$;

(b) $g(0, 0, 0)$.

$$\begin{aligned} g(5,6,10) &= 5^2 + 6 - (3 \times 10) = 1 \\ g(0,0,0) &= 0. \end{aligned}$$

2.3 Partial differentiation

Given a function of two variables, for example

$$z = f(x, y),$$

we can determine two first–order derivatives. The *partial derivative of f with respect to x* is written as

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x,$$

and is found by differentiating f with respect to x, with y held constant.

Similarly, the *partial derivative of f with respect to y* is written as

$$\frac{\partial z}{\partial y} \quad or \quad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y,$$

and is found by differentiating f with respect to y, with x held constant.

Examples

Find the first–order partial derivatives of the functions

(a) $f(x,y) = x^2 + y^3$;

Solution

$$\frac{\partial f}{\partial x} = 2x;$$

$$\frac{\partial f}{\partial y} = 3y^2.$$

Find the first–order partial derivatives of the functions

(b) $f(x,y) = x^2y$;

Solution

$$\frac{\partial f}{\partial x} = 2xy;$$
$$\frac{\partial f}{\partial y} = x^2.$$

Find the first–order partial derivatives of the functions

(c) $f(x, y) = x^2y^3 - 10x$.

Solution

$$\frac{\partial f}{\partial x} = 2xy^3 - 10;$$
$$\frac{\partial f}{\partial y} = 3x^2y^2.$$

In general, when we differentiate a function of two variables, the thing we end up with is itself a function of two variables.

This suggests the possibility of differentiating a second time.

In fact, there are four *second–order partial derivatives*.

We write:

$$\frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx}$$

for the function obtained by differentiating twice with respect to x,

$$\frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

for the function obtained by differentiating twice with respect to y;

$$\frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{yx}$$

for the function obtained by differentiating first with respect to x and then with respect to y, and

$$\frac{\partial^2 z}{\partial x \partial y} \quad \text{or} \quad \frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{xy}$$

for the function obtained by differentiating first with respect to y and then with respect to x.

Find expressions for the second–order partial derivatives f_{xx}, f_{yy}, f_{yx} and f_{xy} for the functions

(a) $f(x,y) = x^2 + y^3$;

(b) $f(x,y) = x^2y$.

Examples

For part (a), we know that

$$f_x = 2x \quad \text{and} \quad f_y = 3y^2.$$

Thus

$$\begin{aligned} f_{xx} &= 2; \\ f_{yy} &= 6y; \\ f_{xy} &= 0; \\ f_{yx} &= 0. \end{aligned}$$

Examples

For part (b), we know that

$$f_x = 2xy \quad \text{and} \quad f_y = x^2.$$

Thus

$$\begin{aligned} f_{xx} &= 2y; \\ f_{yy} &= 0; \\ f_{xy} &= 2x; \\ f_{yx} &= 2x. \end{aligned}$$