Chapter 1

Linear and quadratic functions

What we'll cover...

- What **are** linear/quadratic functions?
- The role of the **gradient** and **intercept**
- **Graphs** of linear/quadratic functions
- Finding the **equation** of a line
- **Solving** linear equations
- **Simultaneous** linear equations
- **Linear programming**
- **Solving** quadratics using various methods
- Chapters 1 and 2 from the course text would be useful here

1.1 Linear functions

Linear functions occur often in accounting and finance.

You should already be familiar with linear functions from your GCSE maths courses. For example,

$$y = 3x + 4$$

is a linear function of x which, when operated, gives us y.

We know this is a linear function because when we produce a graph of this function, we get a straight line.

For example, when $x = 2$ we get

$$y = 3 \times 2 + 4 = 6 + 4 = 10,$$

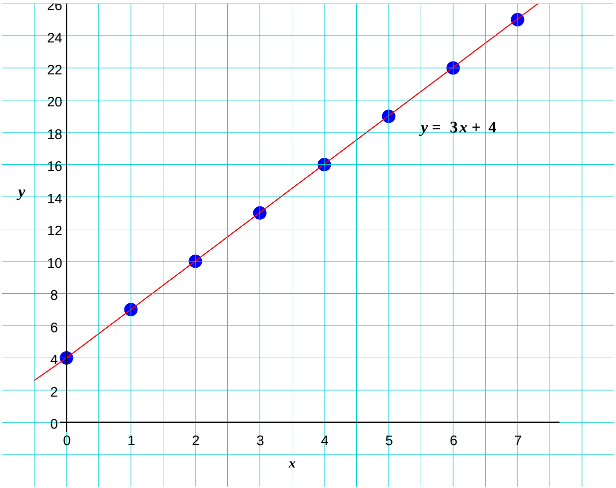
bearing in mind the rules of "BIDMAS": **B**rackets, **I**ndices, **D**ivision, **M**ultiplication, **A**ddition, **S**ubtraction.

Similarly, for other values of x:

x	0	1	**2**	3	4	5	6	7
y	4	7	**10**	13	16	19	22	25

1.1 Linear functions

Plotting *x* against *y* and joining up the points gives the graph shown below.



1.1 Linear functions

Simple linear functions have an **intercept** and a **gradient**.

- Intercept: point at which the line would cut through the y–axis – in A&F, often represents *fixed costs*
- Gradient: tells us how steep the line is, as well as the direction of the line – in A&F, often represents *cost per unit made/sold*

Generally, the linear function

$$y = mx + c$$

has intercept c and gradient m.

So, our old friend

$$y = 3x + 4$$

has gradient 3 and intercept 4 (we saw the intercept on the graph).

For every unit increase in x, y increases by 3 units!

Notice that the gradient is always the number that is “stuck” to the x; that is, the gradient is the x **coefficient**.

Notice also that the highest power in x is a 1 – because $x^1 = x$. There are no terms in x^2 or x^3. Thus, we know we have a linear function.

The intercept is the “other” number, and is sometimes called the **constant**.

As we shall shortly see, if the x coefficient is positive, then the graph has an uphill slope; if the x coefficient is negative, then the graph has a downhill slope.

The bigger the coefficient, the steeper the slope, no matter which direction the slope goes in!

We will come back to finding the equation of a line shortly.

If we set our linear function equal to a number/constant, then we have a linear **equation**.

Solve each of the following linear equations for x and t respectively.

1. $\dfrac{6}{x+4} = 4;$

2. $\dfrac{-4t+4}{5t-6} = -8.$

Solution to (1)

$$\begin{aligned}
\frac{6}{x+4} &= 4 \\
6 &= 4(x+4) \\
6 &= 4x+16 \\
6-16 &= 4x \\
-10 &= 4x \\
x &= -\frac{10}{4} = -2.5.
\end{aligned}$$

Solution to (2)

$$\begin{aligned}
\frac{-4t+4}{5t-6} &= -8 \\
-4t+4 &= -8(5t-6) \\
-4t+4 &= -40t+48 \\
-4t+40t &= 48-4 \\
36t &= 44 \\
t &= \frac{44}{36} = \frac{11}{9} = 1\frac{2}{9} = 1.222
\end{aligned}$$

1.1.2 Finding the equation of a line

Suppose we are told that the gradient of a line is 6. We are also told that the line passes through the point $(3, 25)$. What is the equation of this line?

We are told that $m = 6$, and so we have

$$y = 6x + c.$$

So now we just need to find c! We are also told that the line passes through the point $(3, 25)$, i.e. the point where $x = 3$ and $y = 25$.

Substituting these values into the above equation gives

$$\begin{aligned} 25 &= (6 \times 3) + c \qquad \text{i.e.} \\ 25 &= 18 + c. \end{aligned}$$

So obviously $c = 7$, giving $y = 6x + 7$.

1.1.2 Finding the equation of a line

Suppose we are told that a line passes through the points $(1, 3)$ and $(5, 35)$. What is the equation of the line?

Again, we need to find the gradient, *m*, and the intercept, *c*. We know that

$$\begin{aligned} \text{Gradient} &= \frac{\text{Change in } y}{\text{Change in } x} \\ &= \frac{35 - 3}{5 - 1} \\ &= \frac{32}{4} = 8. \end{aligned}$$

1.1.2 Finding the equation of a line

So now we have

$$y = 8x + c,$$

and we just need to find c.

We know that the line passes through $(1, 3)$, and so substituting $x = 1$ and $y = 3$ into the above equation gives

$$\begin{aligned} 3 &= (8 \times 1) + c \\ 3 &= 8 + c \qquad \text{i.e.} \\ 3 - 8 &= c \qquad \text{and so} \\ -5 &= c. \end{aligned}$$

So the equation of the line is $y = 8x - 5$.

Example 1.1

The United Nations believes the annual consumption of rice in India (y kilograms per household) is a linear function of the unit cost (x US $).

They also know that, when the unit cost of rice is $12, the annual consumption of rice is 40.8kg per household; when this unit cost doubles, the corresponding consumption decreases to 21.6kg per household.

(a) Using this information, obtain the linear function for demand in terms of cost.

(b) Find the unit cost of rice if the demand is 25kg per household.

Example 1.1: Solution to part (a)

$$x \;:\; \text{Cost (\$ US)} \qquad \text{and}$$
$$y \;:\; \text{Consumption (kg)}$$

We are also told that

- when $x = 12$, $y = 40.8 \longrightarrow (12, 40.8)$,
- when $x = 24$, $y = 21.6 \longrightarrow (24, 21.6)$; thus

$$\text{Gradient} = \frac{40.8 - 21.6}{12 - 24} = -1.6.$$

Since we are asked to find the *linear* demand function, we know that $y = mx + c$ (where $m = -1.6$). Taking the point $(12, 40.8)$:

$$\begin{aligned} 40.8 &= -1.6 \times 12 + c \\ 40.8 + 1.6 \times 12 &= c \\ 60 &= c. \end{aligned}$$

So the linear demand function is $y = -1.6x + 60$.

Example 1.1: Solution to part (b)

If demand is 25kg per household, then $y = 25$. Thus

$$\begin{aligned} 25 &= -1.6x + 60 \\ 25 - 60 &= -1.6x \\ -35 &= -1.6x \quad \text{so} \\ x &= 21.875, \end{aligned}$$

or \$21.88.

Solve the following linear equations simultaneously; do *not* use a graph.

$$\begin{aligned} 4x + 8y &= 30 \\ 7x - 6y &= 5. \end{aligned}$$

Solution

Let's eliminate the y's.

Multiplying the first equation by 3 and multiplying the second equation by 4 gives:

$$\begin{aligned} 12x + 24y &= 90 \\ 28x - 24y &= 20. \end{aligned}$$

The signs of the y–coefficients are different, so we add:

$$\begin{aligned} 12x + 24y &= 90 \\ \underline{28x - 24y} &= \underline{20} \\ 40x + 0y &= 110 \end{aligned}$$

and so

$$x = \frac{110}{40} = \frac{11}{4} = 2.75.$$

Solution

Substituting $x = 2.75$ into the first equation gives:

$$\begin{aligned} 4 \times 2.75 + 8y &= 30 \\ 11 + 8y &= 30 \\ 8y &= 19 \\ \\ y &= \frac{19}{8} = 2.375. \end{aligned}$$

Example 1.2

A cookie company makes two types of biscuit: *standard* and *deluxe*. Suppose the company makes x batches of standard biscuits and y batches of deluxe biscuits every day.

(a) Each batch of standard biscuits requires 5kg of flour; each batch of deluxe biscuits requires 8kg of flour. Every day, exactly 98kg of flour must be used. Write down a linear equation in x and y for the total amount of flour used each day (in kg).

(b) Each batch of standard biscuits requires 2kg of butter; each batch of deluxe biscuits requires twice as much butter. Every day, exactly 44 kg of butter must be used. Write down a linear equation in x and y for the total amount of butter used each day (in kg).

(c) How many batches of standard and deluxe biscuits can the company make each day?

Example 1.2: Solution to (a) and (b)

It often helps if you draw up a table:

	Flour (kg)	**Butter (kg)**
Standard (x)	5	2
Deluxe (y)	8	4
Total	98	44

Then for flour (part (a)), we would have

$$5x + 8y = 98.$$

For butter (part (b)), we have

$$2x + 4y = 44.$$

Example 1.2: Solution to (c)

If we keep the flour equation the same, but multiply the butter equation by 2, we get:

$$\begin{aligned} 5x + 8y &= 98 \\ 4x + 8y &= 88 \end{aligned}$$

Subtracting to eliminate the y's gives:

$$\begin{aligned} 1x + 0y &= 10 \\ x &= 10 \end{aligned}$$

Example 1.2: Solution to (c)

Substituting $x = 10$ into the flour equation gives

$$\begin{aligned} 5 \times 10 + 8y &= 98 \\ 50 + 8y &= 98 \\ 8y &= 48 \\ y &= 6. \end{aligned}$$

So, each day, the company must make 10 batches of standard biscuits and 6 batches of deluxe biscuits.

1.2 Quadratic functions

As you might imagine, not everything in real–life can be represented by a straight line!

At best, the linear functions we have looked at so far – such as linear cost functions and linear functions for profit – are simplifications.

With this in mind, in this section we will consider the role of **non–linear functions**.

In particular, we will think about polynomial functions where the power in x is greater than 1 (giving a non–linear graph).

The simplest case here is the **quadratic function**, where the highest power of x in our polynomial is 2.

That is, at most we have an x^2 term. For example,

$$y = 5x^2 - 2x + 6$$

is a quadratic, since the highest power of x is 2. The polynomial

$$y = 2x^4 - 3x^2 + 9$$

is not a quadratic, since the highest power of x here is 4 (in fact, this is a **quartic**).

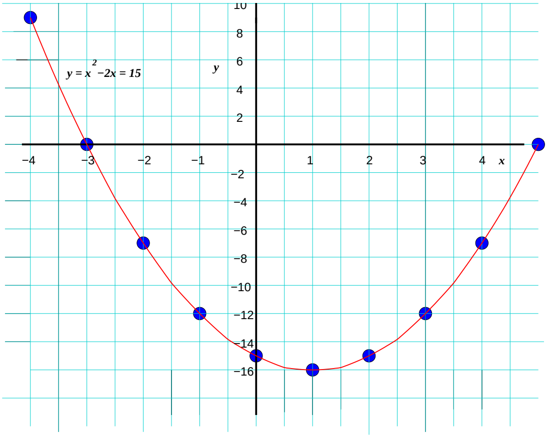
1.2.1 Plotting quadratic functions

In the space below, plot the graph of the function

$$y = x^2 - 2x - 15.$$

Solution

x	–4	–3	–2	–1	0	1	2	3	4	5
y	9	0	–7	–12	–15	–16	–15	–12	–7	0



Example 1.5

The estates manager of a zoo has 100 metres of fencing to construct a rectangular enclosure for some West African camels.

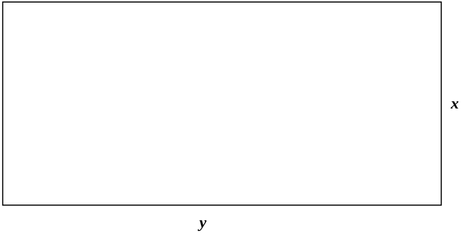
(a) Formulate a non–linear function for the area of the enclosure in terms of its width, x metres.

(b) Produce a plot of your non–linear function in (a) (see overleaf).

(c) Using your plot in (b), what length x would give the optimal area for the camel enclosure? What area would this give?

Example 1.5: Solution to part (a)

Consider the diagram:



The area is obviously $x \times y = xy$, but the question wants the solution in terms of x only.

Example 1.5: Solution to part (a)

Now

$$\text{Perimeter} = x + x + y + y = 2x + 2y = 100.$$

Therefore we can write

$$\begin{aligned} 2y &= 100 - 2x \qquad \text{i.e.} \\ y &= 50 - x. \end{aligned}$$

So now

$$\begin{aligned} \text{Area} &= x \times y \\ &= x(50 - x) \\ &= 50x - x^2. \end{aligned}$$

Example 1.5: Solution to part (b)

Example 1.5: Solution to part (c)

From the graph, we can see that a length of $x = 25m$ would give the largest area for the camel enclosure.

This would give an area of

$$\begin{aligned} \text{Area} &= 50 \times 25 - 25^2 \\ &= 625\text{m}^2. \end{aligned}$$

1.2.1 Plotting quadratic functions

As with linear functions, it would be useful to be able to plot or sketch a non–linear function without having to draw up a full table of results.

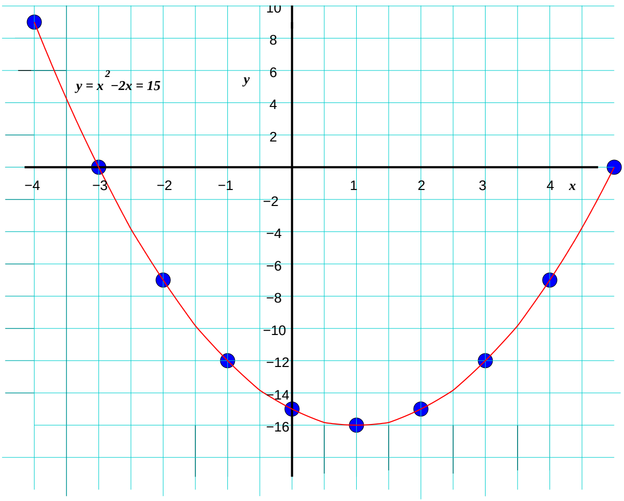
For a quadratic function, it might be useful to know where the curve cuts into the x–axis, as well as the y–intercept of the curve.

Consider the quadratic from the earlier example: $y = x^2 - 2x - 15$. We know the y–intercept occurs when $x = 0$, that is

$$y = 0^2 - 2 \times 0 - 15 = -15,$$

and this can be seen in our plot of this function.

1.2.1 Plotting quadratic functions



1.2.1 Plotting quadratic functions

As with linear functions, it would be useful to be able to plot or sketch a non–linear function without having to draw up a full table of results.

For a quadratic function, it might be useful to know where the curve cuts into the x–axis, as well as the y–intercept of the curve.

Consider the quadratic from the earlier example: $y = x^2 - 2x - 15$. We know the y–intercept occurs when $x = 0$, that is

$$y = 0^2 - 2 \times 0 - 15 = -15,$$

and this can be seen in our plot of this function.

However, how can we tell, without drawing up a full table of results, where the curve will cut the x–axis?

The curve cuts the x–axis when $y = 0$, that is when

$$x^2 - 2x - 15 = 0,$$

and so we would need to solve this equation for x. However, this is an example of a **quadratic equation**, and we haven't considered how to solve these yet...

We can visualise the solution(s) for x in the equation

$$ax^2 + bx + c = 0$$

by looking at a plot of the quadratic and noting where the curve intersects/touches the line $y = 0$, that is, the x–axis.

However, the accuracy of our solutions obtained in this way will depend on the accuracy of our graph.

We would also need to know what range of x–values to draw the graph over.

Two approaches for solving quadratics a bit more mathematically will now be considered.

1.2.2 Quadratic equations

In both, we will assume the general form of a quadratic as

$$y = ax^2 + bx + c;$$

the **discriminant**, D, is given by

$$D = b^2 - 4ac,$$

and has important properties in terms of how we classify the solutions to our quadratic.

1.2.2 Quadratic equations: Factorisation

If the discriminant is a perfect square – that is, if the square root of D is an integer – then we can solve our quadratic by the method of **factorisation**.

Consider the following quadratic equations:

1. $x^2 - 2x - 15 = 0$
2. $x^2 + 5x + 2 = 0$
3. $x^2 - 8x + 12 = 0$
4. $x^2 - 5x - 36 = 0$
5. $2x^2 - 7x + 5 = 0$

(a) For each, find the discriminant D.

(b) Which equations can be solved by factorisation?

(c) Solve the equations identified in (b) by factorisation.

Solution to (a)

1. $x^2 - 2x - 15 = 0$

$$D = (-2)^2 - (4 \times 1 \times -15) = 64$$

2. $x^2 + 5x + 2 = 0$

$$D = 5^2 - (4 \times 1 \times 2) = 17$$

3. $x^2 - 8x + 12 = 0$

$$D = (-8)^2 - (4 \times 1 \times 12) = 16$$

Solution to (a)

4. $x^2 - 5x - 36 = 0$

$$D = (-5)^2 - (4 \times 1 \times -36) = 25 + 144 = 169$$

5. $2x^2 - 7x + 5 = 0$

$$D = (-7)^2 - (4 \times 2 \times 5) = 9$$

Solution to (b)

Note that the discriminant for Equation (1) was $D = 64$:

$$\sqrt{64} = 8,$$

which is an integer value; thus ***D* is a perfect square**, and so we can solve this equation by factorisation.

In fact, we can solve all but equation (2) by factorisation – equation 2 is the only one where the discriminant is *not* a perfect square.

Solution to (c)

Equation (1):

Sinced D was a perfect square, we can factorise our quadratic; in fact,

$$x^2 - 2x - 15 = (x - 5)(x + 3).$$

How did we do this?

- The first term inside each pair of brackets must be x;
- The other two numbers must add together to give the x–coefficient (e.g. $-5 + 3 = -2$)...
- ...and must multiply to give the constant (e.g. $-5 \times 3 = -15$)

Solution to (c)

We can check our factorisation by expanding the brackets using the **F-O-I-L** rule, standing for "**F**irst two", "**O**utside two", "**I**nside two", "**L**ast two":

$$\begin{aligned}
\textbf{F} &: \quad x \times x = x^2 \\
\textbf{O} &: \quad x \times 3 = 3x \\
\textbf{I} &: \quad -5 \times x = -5x \\
\textbf{L} &: \quad -5 \times 3 = -15,
\end{aligned}$$

giving

$$x^2 + 3x - 5x - 15 = x^2 - 2x - 15,$$

which is exactly what we started with!

Solution to (c)

So now we know that

$$x^2 - 2x - 15 = (x-5)(x+3) = 0,$$

we know that either $(x-5) = 0$ or $(x+3) = 0$. So we have

$$\begin{aligned} x - 5 = 0 \quad &\text{or} \quad x + 3 = 0 \\ x = 5 \quad &\text{or} \quad x = -3, \end{aligned}$$

and these are the values we see on the graph on page 19.

Solution to (c)

Equation (3):

$$\begin{aligned} x^2 - 8x + 12 &= 0 \\ (x-6)(x-2) &= 0, \end{aligned}$$

so $x = 6$ or $x = 2$.

Solution to (c)

Equation (4):

$$\begin{aligned} x^2 - 5x - 36 &= 0 \\ (x-9)(x+4) &= 0, \end{aligned}$$

so $x = 9$ or $x = -4$.

Equation (5):

$$\begin{aligned} 2x^2 - 7x + 5 &= 0 \\ (2x-5)(x-1) &= 0, \end{aligned}$$

so $(2x - 5) = 0$ or $(x - 1) = 0$. From the first:

$$2x = 5 \rightarrow x = 2.5,$$

and also $x = 1$.

1.2.2 Quadratic equations: formula

If we cannot factorise our quadratic, then we can use the **quadratic formula** in order to find the solution(s) to our quadratic equation.

For example, consider equation (2) in the last question:

$$x^2 + 5x + 2 = 0.$$

The discriminant

$$D = 5^2 - 4 \times 1 \times 2 = 17,$$

which is not a perfect square.

1.2.2 Quadratic equations: formula

All is not lost – we can use the quadratic formula!

- Developed by **Indian mathematicians** around 628AD
- The formula as we know it today was not published until 1637 by a **French mathematician**
- In fact, the formula can *always* be used – whether the discriminant is a perfect square or not!

1.2.2 Quadratic equations: formula

The solution(s) to the equation

$$ax^2 + bx + c = 0,$$

with discriminant $D = b^2 - 4ac$, are given by the formula

$$x = \frac{-b \pm \sqrt{D}}{2a}.$$

1.2.2 Quadratic equations: formula

Consider the quadratic we plotted on page 19:

$$y = x^2 - 2x - 15.$$

The graph showed that when $y = 0$ the solutions were $x = 5$ and $x = -3$.

Note that $a = 1$, $b = -2$ and $c = -15$. Thus the discriminant is

$$D = (-2)^2 - 4 \times 1 \times -15 = 64.$$

So by the formula, we have

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{64}}{2 \times 1} \\ &= \frac{2 \pm 8}{2}. \end{aligned}$$

So the solutions are given by

$$x = \frac{2+8}{2} = \frac{10}{2} = 5,$$

or

$$x = \frac{2-8}{2} = \frac{-6}{3} = -3,$$

exactly the solutions we obtained graphically (and by factorisation)!

1.2.2 Quadratic equations: formula

Solve the quadratic equation

$$x^2 + 5x + 2 = 0.$$

Solution

The discriminant is

$$D = 5^2 - (4 \times 1 \times 2) = 17.$$

Thus, we have

$$x = \frac{-5 \pm \sqrt{17}}{2},$$

giving

$$x = \frac{-5+\sqrt{17}}{2}$$

$$= -0.44$$

or

$$x = \frac{-5-\sqrt{17}}{2}$$

$$= -4.56$$

Example 1.6

A company models its annual profits using the function

$$P(x) = x^2 + 20x - 300,$$

where P represents profits and x is the number of units sold. In 2011 their profits were £167,700. How many units of their product did they sell?

Example 1.6: Solution

We set the equation equal to 167700 and solve for x:

$$\begin{aligned} x^2 + 20x - 300 &= 167700 \\ x^2 + 20x - 300 - 167700 &= 0 \\ x^2 + 20x - 168000 &= 0 \end{aligned}$$

Now the discriminant is

$$D = 20^2 - (4 \times 1 \times -168000) = 672400,$$

and so

$$\begin{aligned} x &= \frac{-20 \pm \sqrt{672400}}{2} \\ &= \frac{-20 \pm 820}{2}. \end{aligned}$$

Example 1.6: Solution

So

$$x = \frac{-20 + 820}{2} = 400$$

or

$$x = \frac{-20 - 820}{2} = -420.$$

Obviously we can't sell a negative number of units, and so $x = 400$.

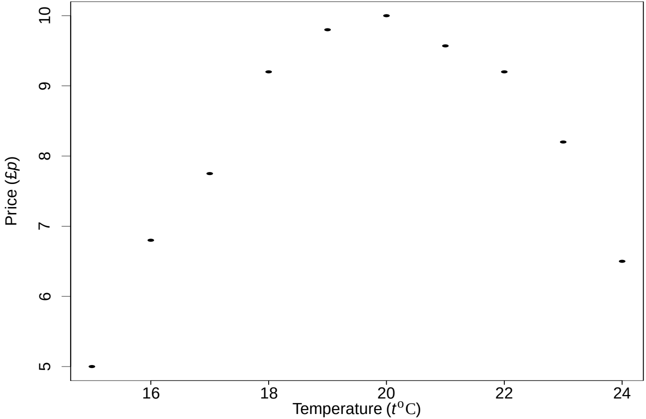
Note the discriminant was a perfect square and so we could have solved this quadratic via factorisation.

Example 1.7

The manager of a vineyard thinks there is a relationship between the average temperature during the grape–growing season ($t^{o}\mathrm{C}$) and the quality of the grapes produced and hence the selling price of resulting bottles of wine (p pounds).

Example 1.7

Over a ten year period the following relationship is revealed:



Example 1.7

(a) Which model would best suit the relationship shown?

- $p = 0.2t + 4.4$
- $p = 0.2t^2 - 8t + 70$
- $p = -0.2t^2 + 8t - 70$

(b) Suppose the manager needs to sell this variety of wine in 2013 for at least £9 per bottle. What is the critical range of temperatures required to meet this price?

Example 1.7: Solution to part (a)

Clearly, it cannot be the first equation, as this is a linear equation and we have a non–linear relationship.

Of the two remaining quadratics, it must be the last one to give the “upside–down” curve we see in the graph (negative coefficient for t^2).

Example 1.7: Solution to part (b)

We set $p = 9$ and solve for t:

$$\begin{aligned} -0.2t^2 + 8t - 70 &= 9 \\ -0.2t^2 + 8t - 70 - 9 &= 0 \\ -0.2t^2 + 8t - 79 &= 0. \end{aligned}$$

Now $a = -0.2$, $b = 8$ and $c = -79$, giving

$$D = 8^2 - (4 \times -0.2 \times -79) = 0.8.$$

Example 1.7: Solution to part (b)

So

$$x = \frac{-8 \pm \sqrt{0.8}}{-0.4},$$

giving

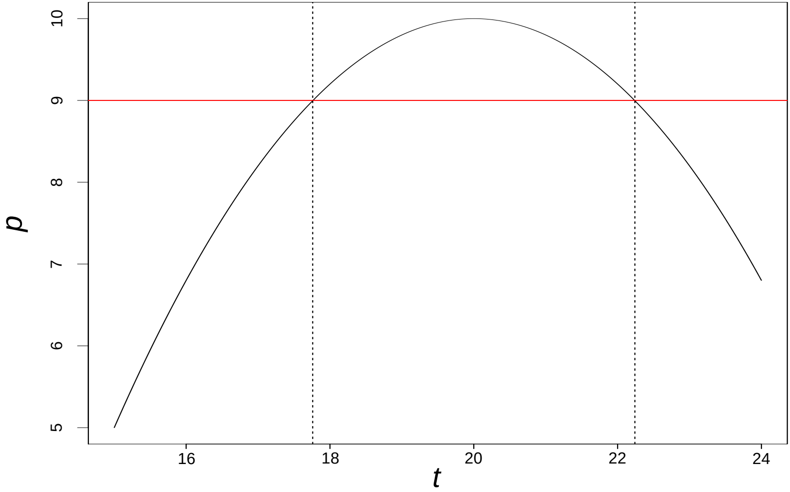
$$x = \frac{-8 + \sqrt{0.8}}{-0.4} = 17.76$$

and

$$x = \frac{-8 - \sqrt{0.8}}{-0.4} = 22.24.$$

So any temperatures between 17.8$^{\text{o}}$C and 22.2$^{\text{o}}$C should give bottles of wine that sell for at least £9.

Example 1.7: Solution to part (b)



Summary

- If D is a perfect square, then we can solve the quadratic by **factorisation**; otherwise, we can use the **formula** (in fact, the formulae will always work!)
- If $D > 0$, our quadratic will have **two real solutions** – that is, the curve will cut the x–axis in two places
- If $D = 0$, then our quadratic will have **repeated solutions** – that is, our graph only just touches the x–axis
- If $D < 0$, then we cannot operate the quadratic formula as we cannot take the square root of a negative – at least, not in the real number system. Actually, such a quadratic has **complex roots** making use of the complex number system, but this goes beyond the scope of ACC1012/53.

1.3 Higher–order polynomials

In more complicated accounting and finance problems, linear and/or quadratic functions might still not be adequate enough to model our situation realistically. Higher–order polynomials can be used, such as **cubics**:

$$y = ax^3 + bx^2 + cx + d,$$

or even **quartics**:

$$y = ax^4 + bx^3 + cx^2 + dx + e;$$

We will return to such functions in Chapter 3 of this course.

Now back to page 8...

1.1.4 Linear programming

Dynamic programming techniques were developed during the Second World War by a group of American mathematicians.

They sought to produce mathematical models of situations in which all the **requirements**, **constraints** and **objectives** were expressed as algebraic equations.

They then developed methods for obtaining the **optimal solution** – the maximum or minimum value of a required function.

1.1.4 Linear programming

In this section, we will attempt to formulate the requirements, constraints and objectives of real–life accounting and finance problems as linear equations.

This branch of dynamic programming is referred to as **linear programming**, for obvious reasons.

Thus, all algebraic expressions will be of the form

$$(\text{a number})x + (\text{a number})y;$$

for example,

$$\text{Profit} = 4x - 3y$$

is a linear equation for profit in terms of x and y, where x might represent our expenditure on advertising and y might be our costs.

1.1.4 Linear programming

Linear programming methods are some of the most widely used methods employed to solve management and economic problems.

They have been applied in a variety of contexts, some of which will be discussed in this chapter, with enormous savings in money and resources.

The first step is to **formulate** a problem as a linear programming problem; the second step is to **solve** the problem.

As well as drawing upon our recent work on linear equations, in this section we also need to think about linear **inequalities**; for example,

$$16x + 18y \quad \leq \quad 25$$

is a linear inequality in x and y. The role of inequalities will become apparent as we work through some examples.

1.1.4 Formulating linear programming problems

The first step in formulating a linear programming problem is to determine which quantities you need to know to solve the problem. These are called the **decision variables**.

The second step is to decide what the **constraints** are in the problem. For example, there may be a limit on resources.

The third step is to determine the obective to be achieved. This is the quantity to be maximised or minimised, that is, *optimised*. The function of the decision variables that is to be optimised is called the **objective function**.

Example 1.3: The chair manufacturer

A manufacturer makes two kinds of chairs – **A** and **B**. Each type of chair has to be processed in two departments – **I** and **II**.

Chair **A** spends 3 hours in department **I** and 2 hours in department **II**. Chair **B** spends 3 hours in department **I** and 4 hours in department **II**.

Example 1.3: The chair manufacturer

The time available in department **I** in any given month is 120 hours, and the time available in department **II** in the same month is 150 hours.

Chair **A** has a selling price of £10 and chair **B** of £12.

Example 1.3: The chair manufacturer

The manufacturer wishes to maximise his income.

How many of each type of chair should be made?

Example 1.3: The chair manufacturer

You'll notice that there's a lot of information given in the question – this is typical of a linear programming problem. Sometimes it's easier to summarise the information given in a table:

Chair	Time in dept. I	Time in dept. II	Selling price
A	3	2	10
B	3	4	12
Time limits	120	150	

Example 1.3: The chair manufacturer

To formulate this linear programming problem, we consider the following three steps:

1. What are the **decision variables**? (i.e. which quantities do you need to know in order to solve the problem?)
2. What are the **constraints**?
3. What is the **objective**?

Step 1: Decision variables

Read through the question and identify the things you'd like to know. You can usually do this by going straight to the last sentence of the question:

"How many of each chair should be made..."

Thus, we'd like to know

- the number of type **A** chairs to make, and
- the number of type **B** chairs to make.

Step 1: Decision variables

These are our decision variables, and are usually denoted with lower case letters. Thus, our decision variables are

$$
\begin{aligned}
x &= \text{number of type } \textbf{A} \text{ chairs made} \qquad \text{and} \\
y &= \text{number of type } \textbf{B} \text{ chairs made.}
\end{aligned}
$$

Step 2: Constraints

This is probably the hardest bit! Consider what could happen in each department.

For example, if we focus on what could happen in department **I**:

Since:	the production of 1 type **A** chair uses 3 hours,
then:	the production of x type **A** chairs takes $3x$ hours.
Similarly:	the production of 1 type **B** chair uses 3 hours,
so:	the production of y type **B** chairs takes $3y$ hours.

Step 2: Constraints

The total time used in department $\mathbf{I}$ is therefore

$$(3x + 3y) \text{ hours.}$$

Since only 120 hours are available in department $\mathbf{I}$, one constraint is

$$\begin{aligned} (3x + 3y) \text{ hours} &\leq 120 \text{ hours}, \qquad \text{or just} \\ (3x + 3y) &\leq 120. \end{aligned}$$

Step 2: Constraints

Considering department **II** in a similar way, we get:

Since:	the production of 1 type **A** chair uses 2 hours,
then:	the production of x type **A** chairs takes $2x$ hours.
Similarly:	the production of 1 type **B** chair uses 4 hours,
so:	the production of y type **B** chairs takes $4y$ hours.

The total time used is therefore

$$(2x + 4y) \text{ hours.}$$

Since only 150 hours are available in department **II**, a second constraint is

$$\begin{aligned}(2x + 4y) \text{ hours} &\leq 150 \text{ hours}, \qquad \text{or just} \\ (2x + 4y) &\leq 150.\end{aligned}$$

Step 2: Constraints

We’re still not done! We can’t make a negative number of chairs, so we also have:

$$\begin{aligned} x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

These are called the **non–negativity constraints**.

Step 3: Objective function

Our objective here is to **maximise income**.

If we make x type **A** chairs, then we get £$10 \times x =$ £$10x$, since each type **A** chair sells for £10.

Similarly, if we make y type **B** chairs, then we get £$12 \times y =$ £$12y$, since each type **B** chair sells for £12.

Step 3: Objective function

The total income is then

$$£Z = £(10x + 12y).$$

The aim is to maximise income, so we'd like to maximise

$$Z = 10x + 12y,$$

where Z is the objective function.

Thus, to summarise, we have the following linear programming problem:

Maximise $Z = 10x + 12y$ subject to the constraints

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

Example 1.4: Replica sports shirts

Sportizus Clothing Company produce replica football shirts and replica rugby shirts for sale on the high street. Each shirt produced goes through a **sewing process** and a **transfer process**.

Each football shirt requires **8** minutes of **sewing time** and **9** minutes for the **transfer process**, whereas rugby shirts each require **5** minutes of **sewing time** and **15** minutes for the **transfer process**.

In any given day, the **total time** available for the sewing process and transfer process is **10** hours and **15** hours respectively.

Example 1.4: Replica sports shirts

To meet current demand, *Sportizus* must produce at least **30** football shirts and **10** rugby shirts each day.

The company sells football shirts and rugby shirts at a **profit** of £22 and £16 respectively.

How many of each type of shirt should Sportizus produce in order to maximise profits?

Example 1.4: Replica sports shirts

Let's start off with a table which summarises the question:

	Sewing (mins)	Transfer (minutes)	Profit (P)
Football	**8**	**9**	**22**
Rugby	**5**	**15**	**16**
Total time	**600**	**900**	

Step 1: Decision variables

The decision variables are the number of football and rugby shirts to make. Let

$$x = \text{number of football shirts to make} \quad \text{and}$$

$$y = \text{number of rugby shirts to make.}$$

Step 2: Constraints

The constraints are:

$$\textbf{sewing} : 8x + 5y \leq 600 \quad \text{and}$$

$$\textbf{transfer} : 9x + 15y \leq 900$$

We also have the **non–negativity contrainits**. However, we are also told that we must make at least 30 football shirts and 10 rugby shirts, giving

$$x \geq 30 \quad \text{and}$$

$$y \geq 10$$

Step 3: Objective function

The aim is to **maximise profit** – call this P.

We know that we make £22 for each football shirt that we make and sell. Since we make (and sell) x football shirts, this will give

$$\pounds 22x \qquad \text{profit.}$$

We also know that we make £16 for each rugby shirt we make and sell. Since we make (and sell) y rugby shirts, this will give

$$\pounds 16y \qquad \text{profit.}$$

Thus total profit is

$$\pounds P = 22x + 16y,$$

Summary

Thus, to summarise:

Maximise $P = 22x + 16y$ subject to the constraints

$$\begin{aligned} 8x + 5y &\leq 600, \\ 9x + 15y &\leq 900, \\ x &\geq 30 \quad \text{and} \\ y &\geq 10. \end{aligned}$$

Solving linear programming problems

In the previous section we discussed how to **formulate** a linear programming problem.

We now consider how to **solve** such problems.

Linear programming problems that involve only two decision variables, x and y, may be solved **graphically** – that is, by drawing lines associated with the constraints, identifying a region of possible solutions and then locating a point in this region that has a particular property.

We will now re–visit the two examples from the previous section.

Example 1.3 revisited (the chair manufacturer)

Recall that the problem posed in **example 1.3** could be summarised in the following way:

Maximise $Z = 10x + 12y$ subject to the following constraints:

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

Example 1.3 revisited (the chair manufacturer)

To find the **feasible region** for this problem (i.e. the region on a graph which satisfies all of our inequalities), we proceed by indicating, on a diagram, the region for which all of the inequalities hold.

Example 1.3 revisited (the chair manufacturer)

The first inequality is $3x + 3y \leq 120$. To show this on a graph, we first need to plot the line $3x + 3y = 120$.

- When $x = 0$, we have

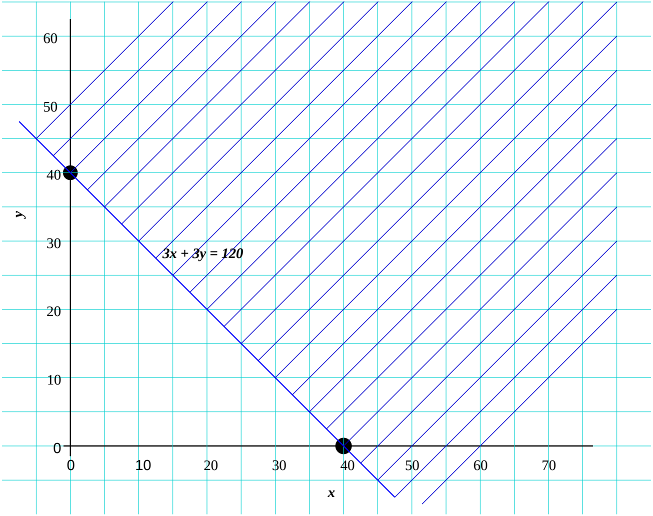
$$\begin{aligned} 3 \times 0 + 3y &= 120 \quad \text{i.e.} \\ 3y &= 120 \quad \text{i.e.} \\ y &= 40. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 3x + 3 \times 0 &= 120 \quad \text{i.e.} \\ 3x &= 120 \quad \text{i.e.} \\ x &= 40. \end{aligned}$$

The points $(0, 40)$ and $(40, 0)$ have been plotted on the graph on page 16, and shading used to illustrate the inequality.

Example 1.3 revisited (the chair manufacturer)



Example 1.3 revisited (the chair manufacturer)

Now consider the second inequality $2x + 4y \leq 150$. Again, to show this on a diagram, we first need to plot the line $2x + 4y = 150$.

- When $x = 0$, we have

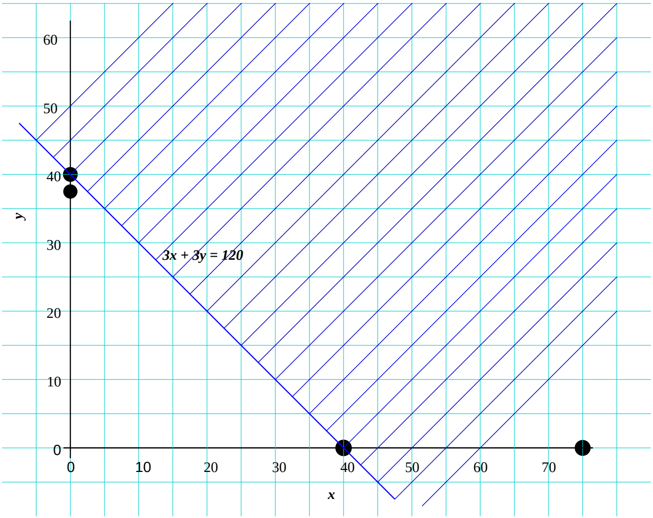
$$\begin{aligned} 2 \times 0 + 4y &= 150 \quad \text{i.e.} \\ 4y &= 150 \quad \text{i.e.} \\ y &= 37.5. \end{aligned}$$

- When $y = 0$, we have

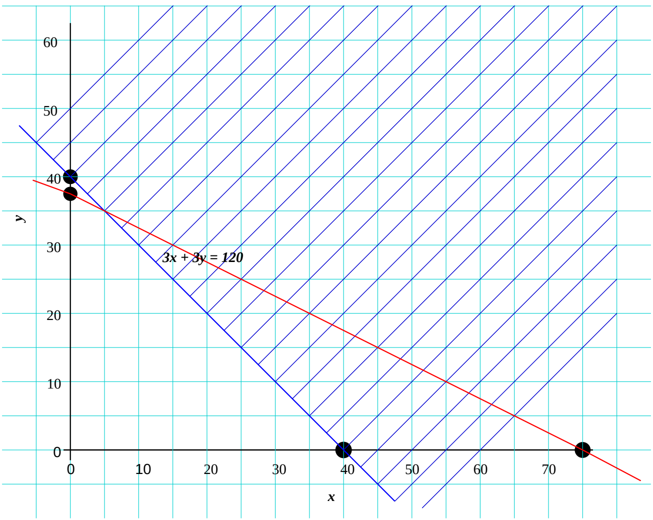
$$\begin{aligned} 2x + 4 \times 0 &= 150 \quad \text{i.e.} \\ 2x &= 150 \quad \text{i.e.} \\ x &= 75. \end{aligned}$$

On the same graph, you should plot the points $(0, 37.5)$ and $(75, 0)$, and shade accordingly.

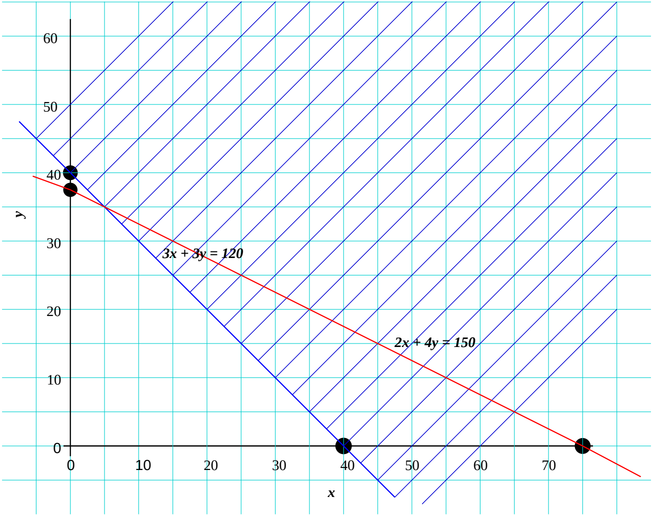
Example 1.3 revisited (the chair manufacturer)



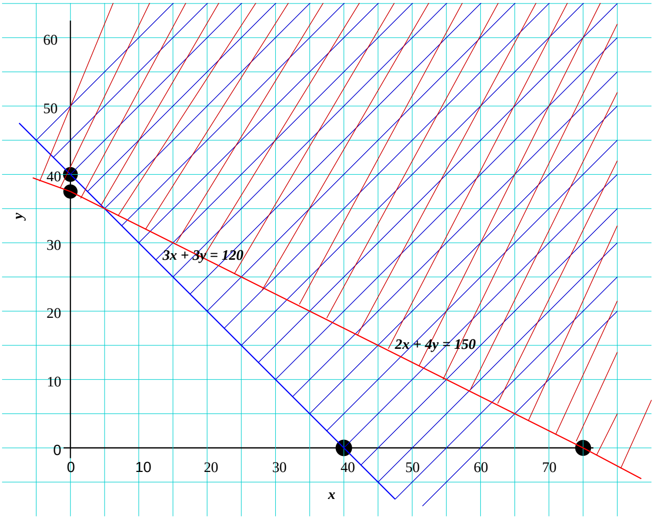
Example 1.3 revisited (the chair manufacturer)



Example 1.3 revisited (the chair manufacturer)

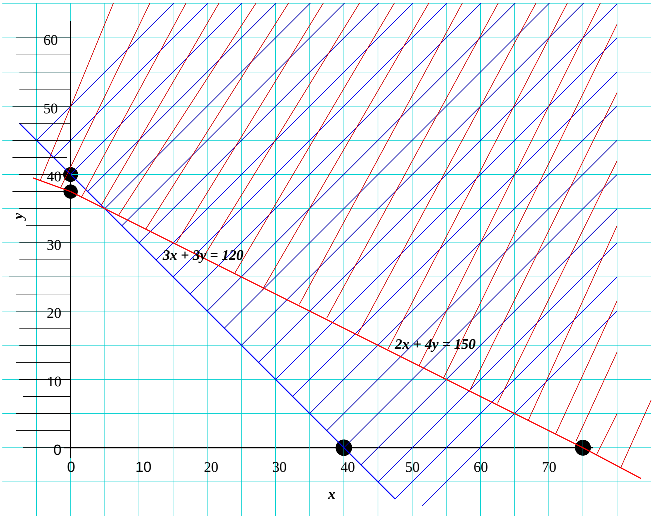


Example 1.3 revisited (the chair manufacturer)

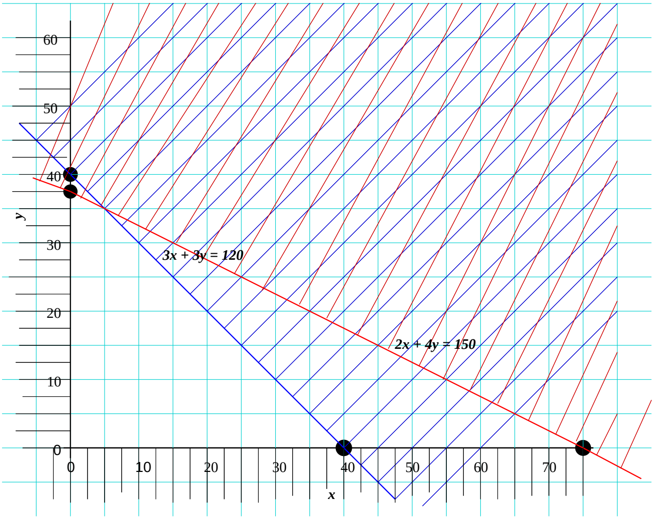


We also need to respect the non–negativity constraints, and so we shade out the negative portion of the graph.

Example 1.3 revisited (the chair manufacturer)



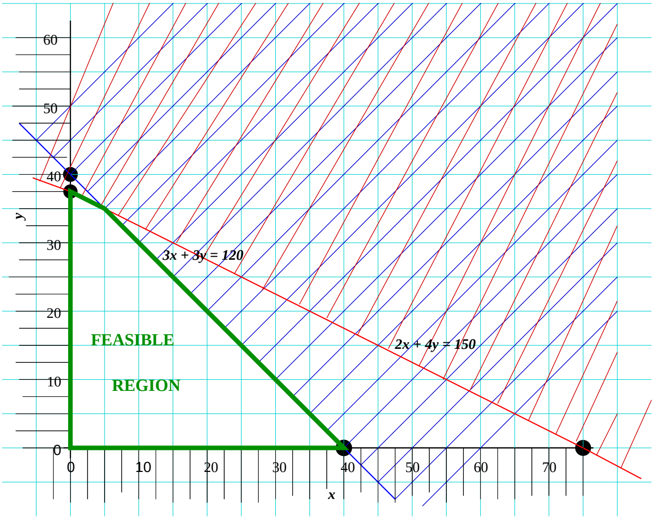
Example 1.3 revisited (the chair manufacturer)



Example 1.3 revisited (the chair manufacturer)

The unshaded region in your graph shows the feasible region associated with our set of inequalities.

Example 1.3 revisited (the chair manufacturer)



Example 1.3 revisited (the chair manufacturer)

The unshaded region in your graph shows the feasible region associated with our set of inequalities.

What we must do now is find the point in that region which meets our objective – i.e. the point in that region which **maximises income**.

One way of doing this is to also plot the **objective function**.

Example 1.3 revisited (the chair manufacturer)

Now our objective function is

$$Z = 10x + 12y,$$

where Z is our income.

When Z takes different values we get a family of **parallel straight lines**.

We need to choose a starting value for Z in order to be able to plot the objective function.

Example 1.3 revisited (the chair manufacturer)

It's often a good idea to try a value which is a multiple of both the coefficients of x and y.

The coefficient of x is 10 and the coefficient of y is 12, so we could try a starting value of $Z = 120$. Thus, the objective function is now

$$10x + 12y = 120.$$

We can plot this line in the same way as before – i.e. consider what happens when x and y are zero.

Example 1.3 revisited (the chair manufacturer)

- When $x = 0$, we have

$$\begin{aligned} 10 \times 0 + 12y &= 120 \quad \text{i.e.} \\ 12y &= 120 \quad \text{i.e.} \\ y &= 10. \end{aligned}$$

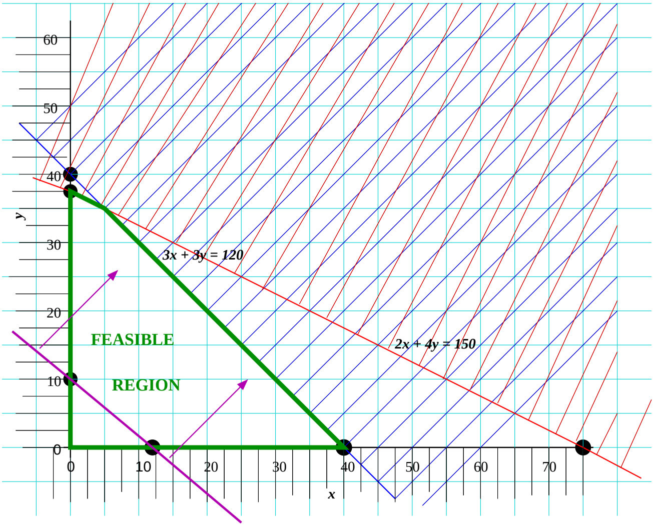
- When $y = 0$, we have

$$\begin{aligned} 10x + 12 \times 0 &= 120 \quad \text{i.e.} \\ 10x &= 120 \quad \text{i.e.} \\ x &= 12. \end{aligned}$$

This line should also plotted on your graph.

- Notice that this line does not give the **optimal income**
- The origin represents zero income, and we want to move as far away from this as possible.

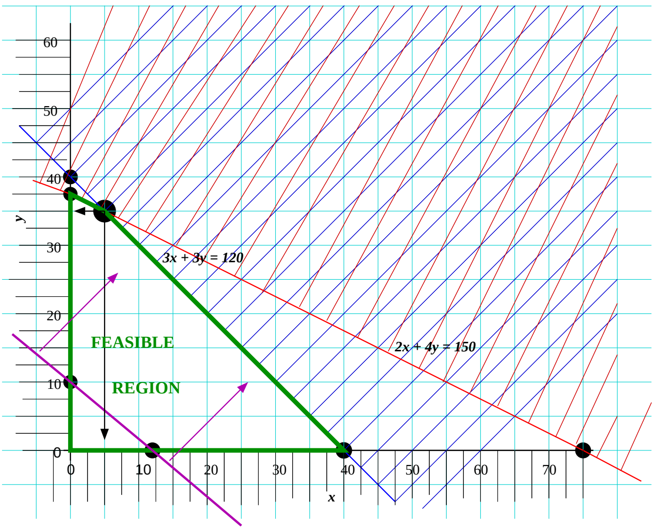
Example 1.3 revisited (the chair manufacturer)



Example 1.3 revisited (the chair manufacturer)

You should see from your graph that the point within the feasible region that is furthest away from the origin, **but parallel to the objective function**, is $\boldsymbol{x = 5}$ and $\boldsymbol{y = 35}$.

Example 1.3 revisited (the chair manufacturer)



Example 1.3 revisited (the chair manufacturer)

You should see from your graph that the point within the feasible region that is furthest away from the origin, **but parallel to the objective function**, is $\boldsymbol{x = 5}$ and $\boldsymbol{y = 35}$.

Thus, to maximise income, we should make **5 type A chairs** and **35 type B chairs**.

This will give an income of

$$\begin{aligned} Z &= 10x + 12y \qquad \text{i.e.} \\ Z &= 10 \times 5 + 12 \times 35 \qquad \text{i.e.} \\ Z &= 50 + 420 \\ &= 470, \end{aligned}$$

i.e. £470.

Example 1.4 revisited (*Sportizus*)

Recall the example about *Sportizus*, a company who produce replica football and rugby shirts. We summarised this problem as follows.

Maximise $P = 22x + 16y$ subject to the following constraints:

$$\begin{aligned} 8x + 5y &\leq 600, \\ 9x + 15y &\leq 900, \\ x &\geq 30 \quad \text{and} \\ y &\geq 10, \end{aligned}$$

where x and y are the number of football and rugby shirts to make, respectively.

Let's now solve this problem graphically.

Example 1.4 revisited (*Sportizus*)

The first inequality is $8x + 5y \leq 600$:

- When $x = 0$, we have

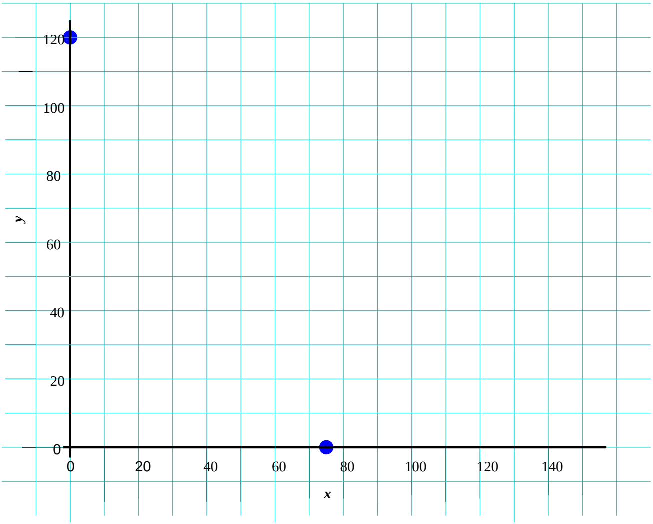
$$\begin{aligned} 5y &= 600 \quad \text{i.e.} \\ y &= 120. \end{aligned}$$

- When $y = 0$, we have

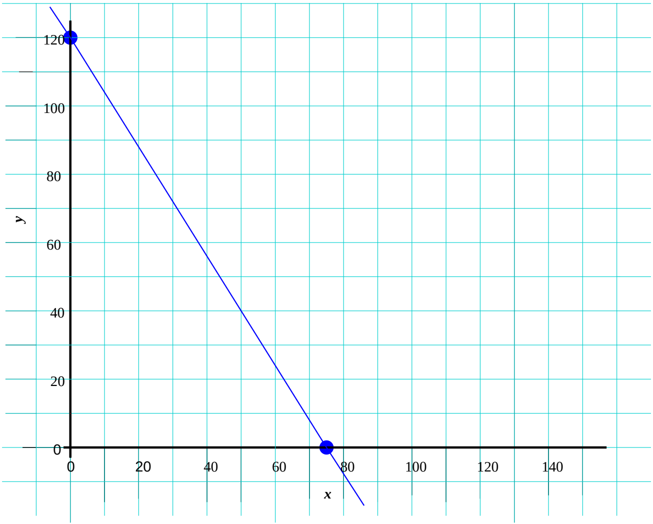
$$\begin{aligned} 8x &= 600 \quad \text{i.e.} \\ x &= 75. \end{aligned}$$

So plot $(0, 120)$, $(75, 0)$.

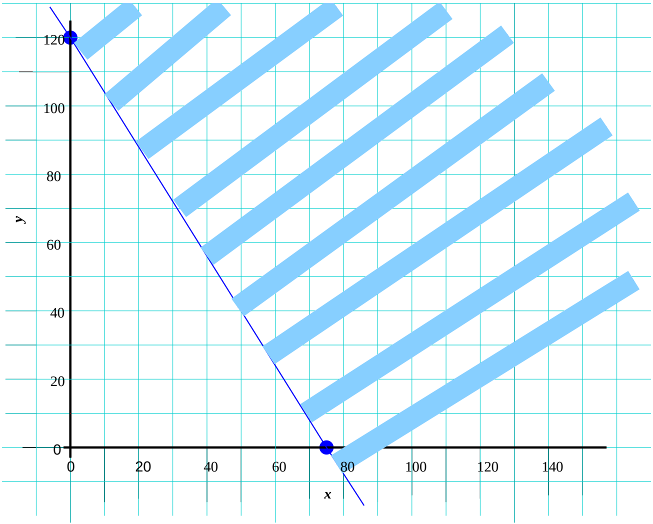
Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)

The second inequality is $9x + 15y \leq 900$:

- When $x = 0$, we have

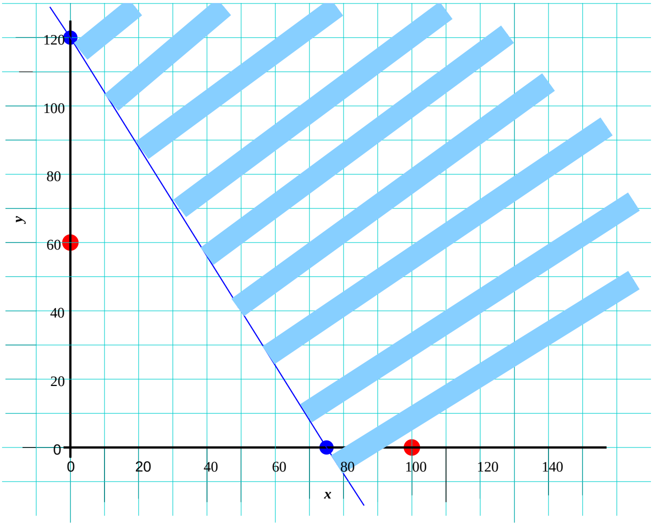
$$\begin{aligned} 15y &= 900 \quad \text{i.e.} \\ y &= 60. \end{aligned}$$

- When $y = 0$, we have

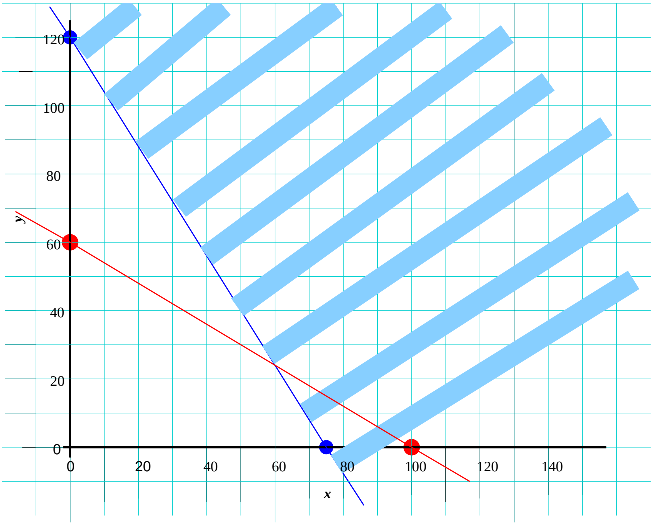
$$\begin{aligned} 9x &= 900 \quad \text{i.e.} \\ x &= 100. \end{aligned}$$

So plot $(0, 60)$ and $(100, 0)$.

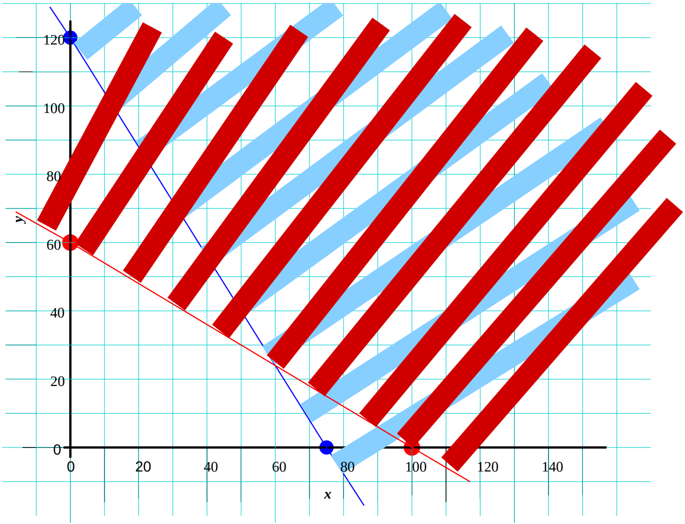
Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)

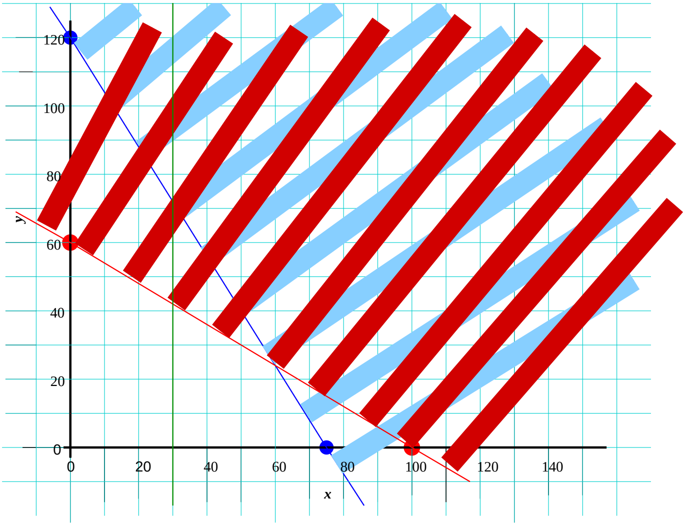


Example 1.4 revisited (*Sportizus*)

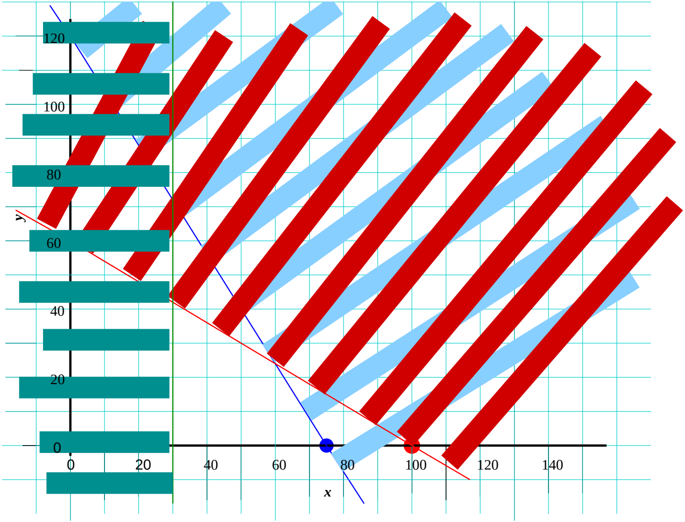
We also require $x \geq 30$ and $y \geq 10$.

Thus, on the graph, you should also shade out all x values less than 30 and all y values less than 10, to leave $x \geq 30$ and $y \geq 10$ *unshaded*.

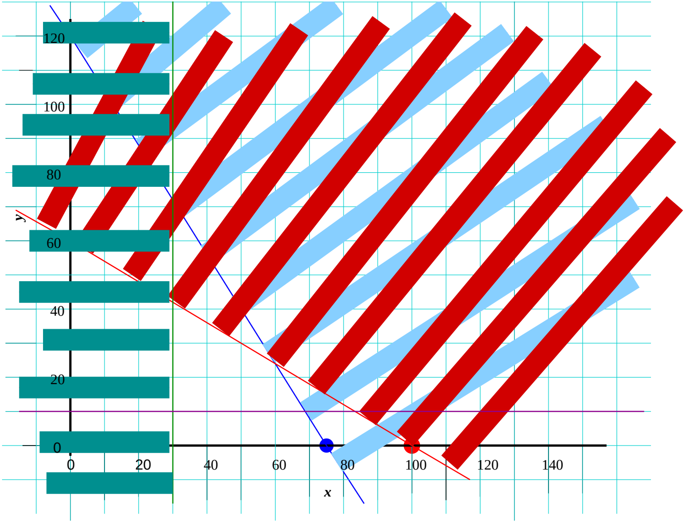
Example 1.4 revisited (*Sportizus*)



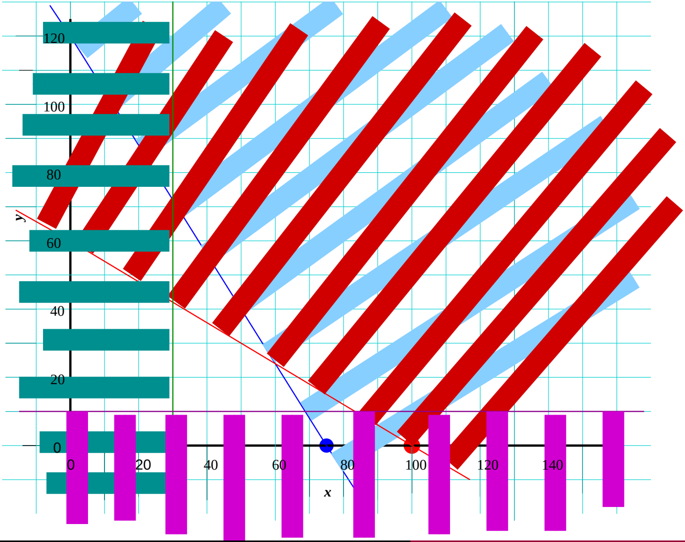
Example 1.4 revisited (*Sportizus*)



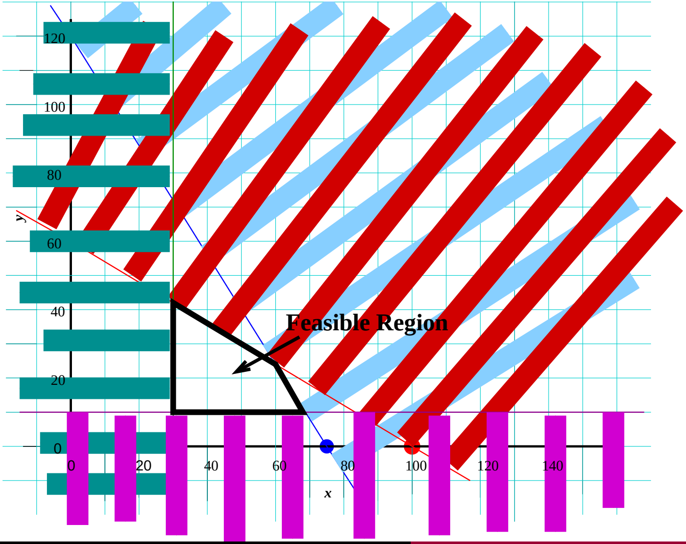
Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)

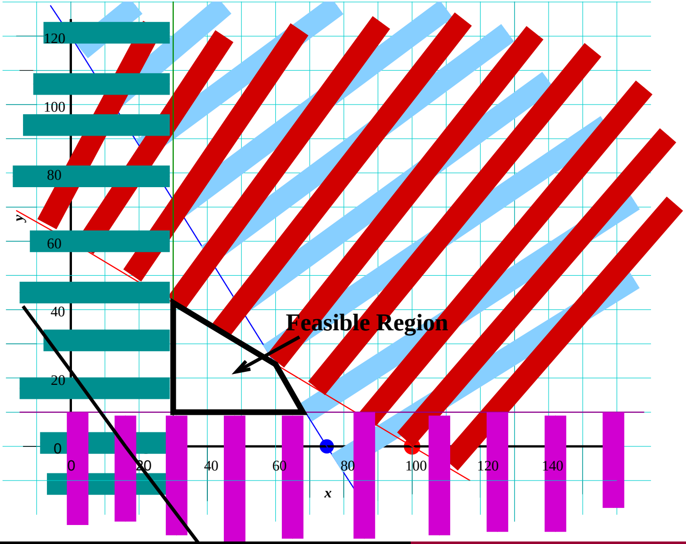
Finally, we need to plot the **objective function**. Here, we have

$$P = 22x + 16y;$$

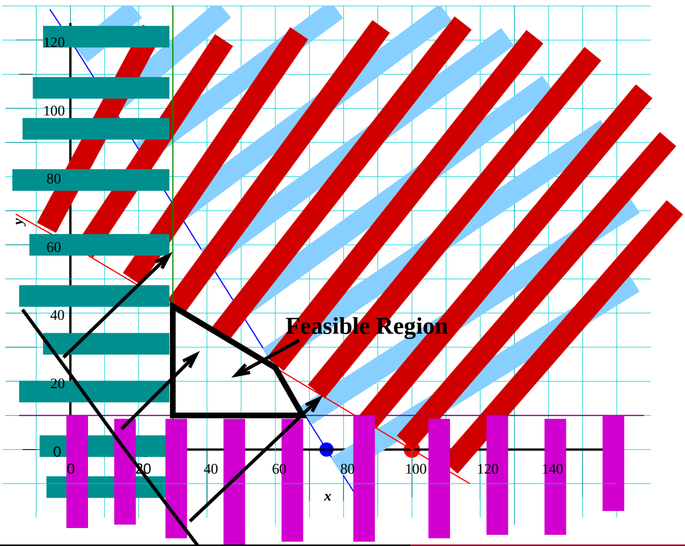
as with the previous example, we need a starting value for P in order to plot this line.

You should notice that starting with $P = 22 \times 16 = 352$ will give $(0, 22)$ and $(16, 0)$, and so we also plot this line on the graph.

Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)



Example 1.4 revisited (*Sportizus*)

To maximise profit, $\boldsymbol{x = 60}$ and $\boldsymbol{y = 24}$, i.e. we should make **60 football shirts** and **24 rugby shirts** each day.

This will give a maximum daily profit of

$$\begin{aligned} P &= 22x + 16y \\ &= 22 \times 60 + 16 \times 24 \\ &= £1704. \end{aligned}$$

Algebraic solutions for linear programming problems

Notice that in both the examples we have looked at so far, the graphical representation indicates that the solution is at the intersection of two lines on the graph.

Thus, we could find also the solution **algebraically**, as the solution to a pair of **simultaneous equations**.

Algebraic solutions for linear programming problems

For example, for *Sportizus*, the solution was at the intersection of the two lines

$$\begin{aligned} 8x + 5y &= 600 \\ 9x + 15y &= 900. \end{aligned}$$

Multyplying the first equation by 3 will allow us to eliminate y on subtraction:

$$\begin{aligned} 24x + 15y &= 1800 \\ \underline{9x + 15y} &= \underline{900} \\ 15x + 0y &= 900, \end{aligned}$$

and so $x = 900/15 = 60$. Substituting this into the first equation gives:

$$\begin{aligned} 480 + 5y &= 600 \\ 5y &= 120 \\ y &= 24. \end{aligned}$$