Chapter 2

Optimisation using calculus

An important topic in many disciplines, including accounting and finance, is the study of how quickly quantities change over time. For example, in order to estimate the future demand for a commodity, we need information about *rates of change*. As we shall see in this chapter, we can also use such information to solve optimisation problems – for example, we might be interested in the number of items that need to be produced in order to maximise profit or minimise loss.

We have already considered rates of change in Chapter 1. In the equation of the straight line

$$y = mx + c,$$

the gradient m represents the rate of change of y with respect to x, and we thought about how to calculate m using the formula

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x}.$$

In Chapter 1 we also considered non–linear functions, focussing primarily on quadratic functions. We will begin Chapter 2 with an examination of how to find the gradient of a non–linear function, before thinking about how this can be used to aid the process of optimisation.

2.1 Differentiation

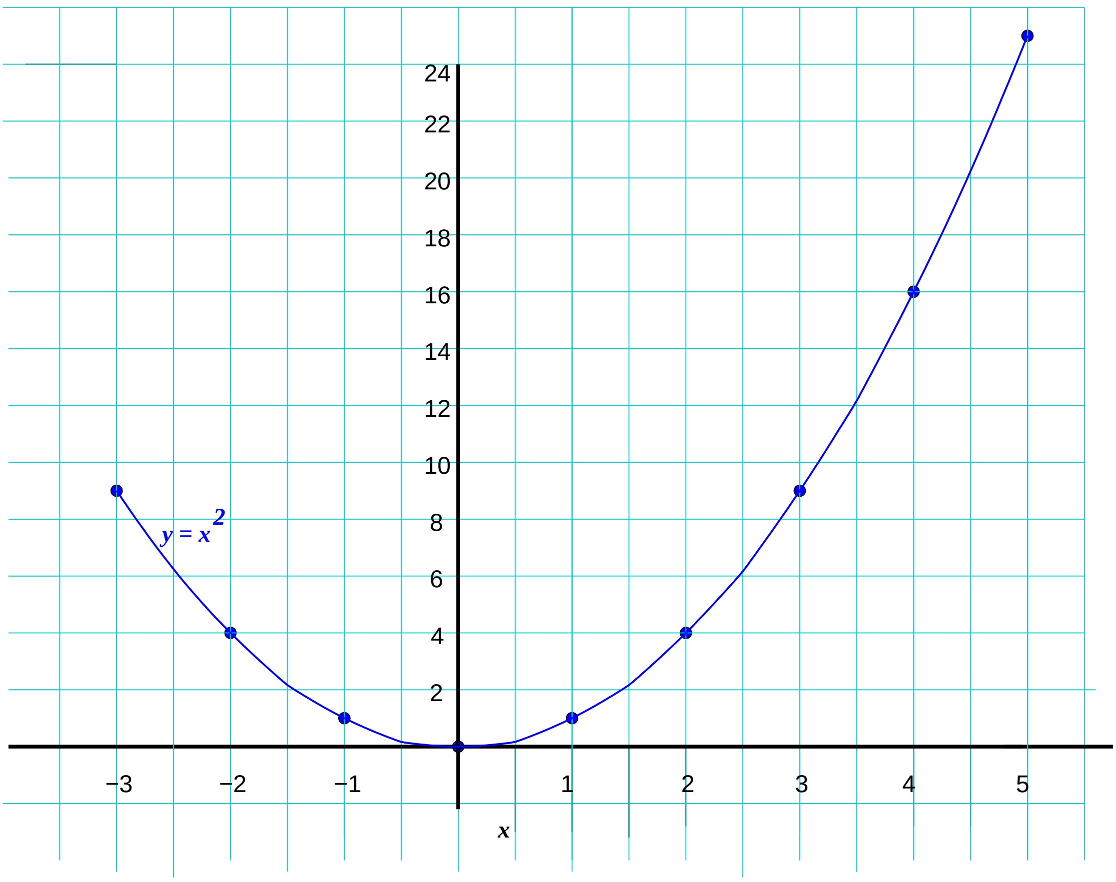
The concept used to describe the rate of change of a function (linear or non–linear) is the *derivative*, which is **the** central concept in mathematical analysis. In this section we will define the derivative of a function, as well as present some of the important rules for calculating it. Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) discovered most of these general rules independently of each other. This initiated the *differential calculus*, which has been the foundation for the development of modern science. It has also been of central importance to the theoretical development of modern economics.

2.1.1 The gradient of a curve

A curve does not have a constant gradient – its direction is continuously changing, and so its gradient will continuously change too. So we look at the gradient of the curve at a particular point on the curve, rather than calculate the gradient of an entire line as with a linear function.

Look at the graph of $y = x^2$ shown below. Let's suppose we are interested in the gradient of this curve at the point $x = 3$. Informally, we can say a couple of things about the gradient at this point:

- The gradient will be positive – as you look at the graph from left to right at this point, the curve goes uphill;
- The graph is symmetric about the point $x = 0$, and so the gradient at $x = -3$ will be the same as that at $x = 3$, just negative;
- As we move from $x = 3$ to $x = 0$, the gradient gets less and less steep, and "flattens out" completely at $x = 0$.



We know how to find the gradient of a straight line from Chapter 1, so we can *approximate* the gradient of the curve at $x = 3$ by drawing a chord on the curve. For example, draw a chord on the curve between the points $x = 3$ and $x = 5$, and find the gradient of this chord *[the co-ordinates will be (3,9) and (5,25)]*:

$$\text{Gradient} = \frac{25-9}{5-3} = \frac{16}{2} = 8.$$

Now decrease the length of the chord and bring it closer to the curve – for example, draw the chord between the points $x = 3$ and $x = 4$ – and find the gradient *[The co-ordinates will be (3,9) and (4,16)]*:

$$\text{Gradient} = \frac{16-9}{4-3} = \frac{7}{1} = 7.$$

Now decrease the length of the chord again and bring it even closer to the curve – draw the chord between the points $x = 3$ and $x = 3.5$ – and find the gradient *[The co-ordinates will be (3,9) and (3.5,12.25)]*:

$$\text{Gradient} = \frac{\qquad\qquad}{\qquad\qquad} =$$

Do this again for $x = 3.25$, $x = 3.1$ and $x = 3.01$, and complete the following table:

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Point of interest	Point 2	Gradient
$(3, 9)$	$(5, 25)$	$\frac{25-9}{5-3} = 8$
$(3, 9)$	$(4, 16)$	$\frac{16-9}{4-3} = 7$
$(3, 9)$	$(3.5, 12.25)$	
$(3, 9)$	$(3.25, 10.5625)$	
$(3, 9)$	$(3.1, 9.61)$	
$(3, 9)$	$(3.01, 9.0601)$	

So it looks like at $x = 3$ the gradient converges to 6. We can generalise this approach algebraically. Suppose the point of interest has co–ordinates (x, x^2) and point 2 has an x co–ordinate close to that of the point of interest, say $x + \delta$. Then the y co–ordinate of point 2 is

$$y = (x + \delta)^2.$$

Thus, the gradient is given by

$$\begin{aligned}
\text{Gradient} &= \frac{\text{Change in } y}{\text{Change in } x} \\
&= \frac{(x+\delta)^2 - x^2}{(x+\delta) - x} \\
&= \frac{(x+\delta)(x+\delta) - x^2}{x + \delta - x} \\
&= \frac{x^2 + 2\delta x + \delta^2 - x^2}{\delta} \\
&= \frac{2\delta x + \delta^2}{\delta} = 2x + \delta.
\end{aligned}$$

Now if we let δ get smaller and smaller, i.e. "tend to zero", we are left with just

$$\text{Gradient} = 2x.$$

So the gradient, or *derivative*, of the curve $y = x^2$ is $2x$. We usually denote this quantity as $\frac{dy}{dx}$, pronounced "dee y by dee x", and so when

$$\begin{aligned}
y &= x^2 \\
\frac{dy}{dx} &= 2x.
\end{aligned}$$

.

We can now use this derivative to find the gradient of the curve at any point x. For example, when $x = 3$, the gradient is

$$\frac{dy}{dx} = 2x = 2 \times 3 = 6,$$

exactly the answer we found from first principles in the table on page 41!

What is the gradient of the curve $y = x^2$ at the points $x = 4$, $x = 1$, $x = 0$ and $x = -4$? Do your answers make sense in relation to the graph of this function?

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2.1.2 Some general results for polynomials

In the previous section we found that, when

$$\begin{aligned} y &= x^2, \\ \frac{dy}{dx} &= 2x. \end{aligned}$$

Notice that the power of x has been "brought down", in front of the x, and the power itself has reduced by one ($2x$ is actually $2x^1$). In general, when

$$\begin{aligned} y &= x^n, \\ \frac{dy}{dx} &= nx^{n-1}. \end{aligned}$$

More generally still, if

$$\begin{aligned} y &= kx^n, \qquad \text{then} \\ \frac{dy}{dx} &= nkx^{n-1}. \end{aligned}$$

Examples

Find $\frac{dy}{dx}$ for each of the following:

1. $y = x^3$

2. $y = x^5$

3. $y = 2x^3$

4. $y = 4x^3 - 7x^2 + 2x$

5. $y = 5x^2 + 2$

6. $y = \sqrt{x} + 1$

7. $y = \frac{1}{x}$

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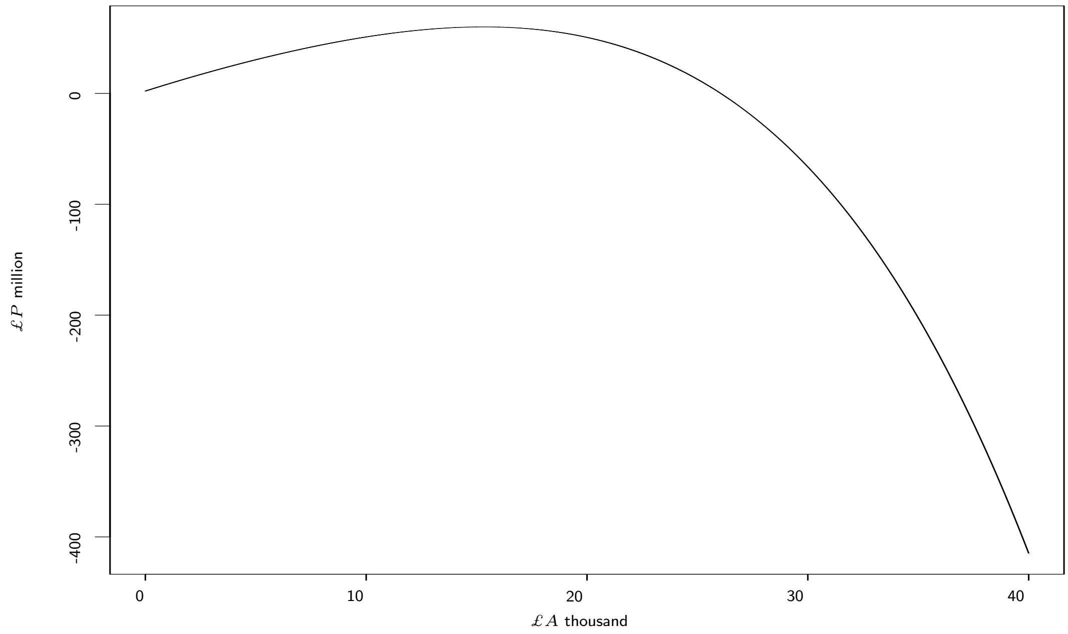
Example 2.1

The managing director of a company is interested in modelling the relationship between her company's annual profits (£P million) and their annual advertising budget (£A thousand). The graph below shows how the managing director believes P changes with A.



(a) You work as part of a team of business analysts for this company. A colleague proposes the following three profit functions:

* $P = 80 - 4.3A$
* $P = 2.1 + 13.5A + 0.5A^2$
* $P = 2.1 + 6.3A - 0.13A^2 + 0.0008A^3 - 0.0002A^4$

Which of these do you think would be most appropriate in this situation? Why?

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(b) For the profit function you have chosen in part (a), find $\dfrac{dP}{dA}$.

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(c) Use your answer to part (b) to find the gradient of your profit function when the company spends (i) £10,000; (ii) £15,250; (iii) £20,000 and (iv) 40,000 on advertising each year. Do these values match up with what you see in the graph?

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(d) What will be the value of $\frac{dP}{dA}$ when the company optimises their advertising budget (i.e. maximises profit)?

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Example 2.2

A small firm employs five machine operators. For a particular contract, the firm believes it will produce $Q = \frac{1}{2}\sqrt{M}$ units of a commodity, where M machine operators are used. The cost, per operator, is €40 and the price obtained per unit is €160.

(a) Formulate a non–linear function for the firm's profit, π, in terms of M only.

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(b) Find $\dfrac{d\pi}{dM}$, and solve $\dfrac{d\pi}{dM} = 0$ for M. Comment.

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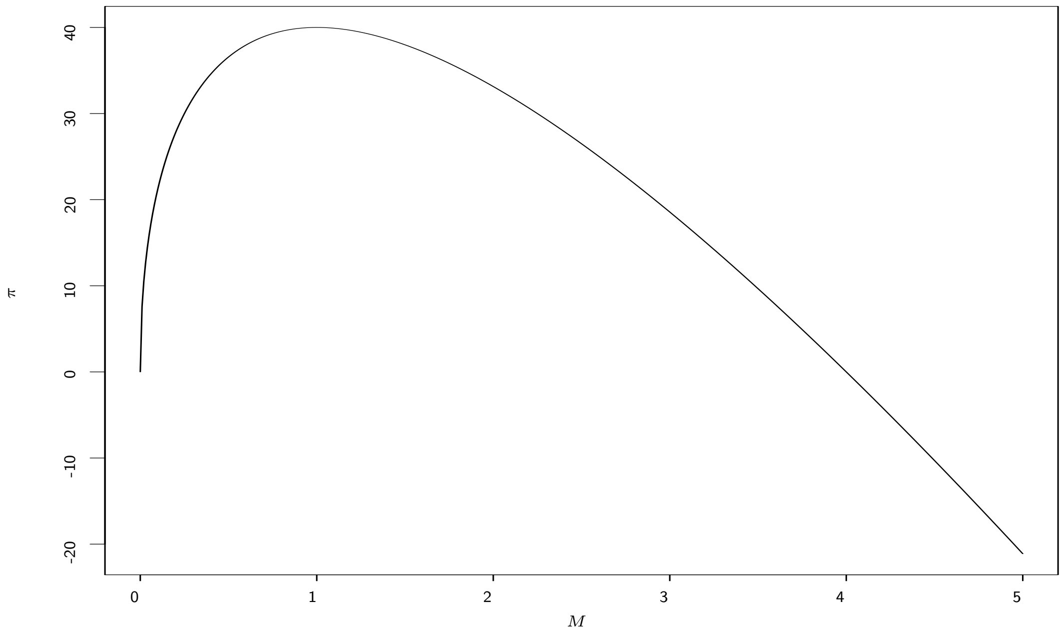
(c) Shown below is a plot of π against M. Can you see how this corresponds to your answer to part (b)?



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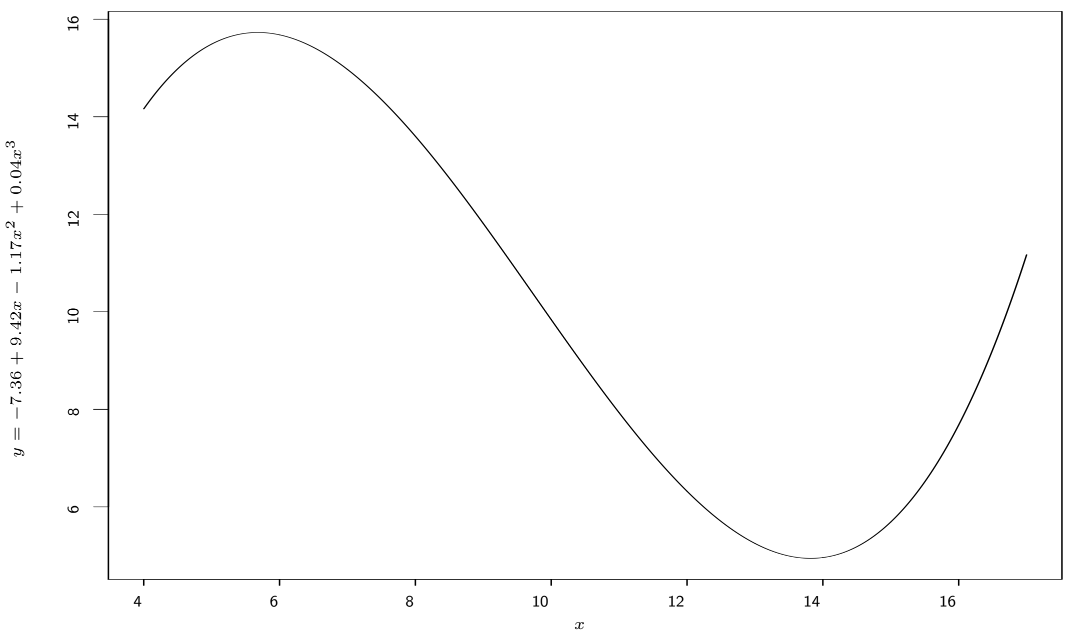
2.2 Optimisation using differentiation

Suppose a company believes there is a non–linear relationship between it's monthly advertising budget (£x thousand) and their monthly profit (£y thousand). In particular, their analyst believes that the following cubic function explains how y varies with x:

$$y = -7.36 + 9.42x - 1.17x^2 + 0.04x^3.$$

The company's monthly advertising budget cannot exceed £17,000; their current contractual arrangements for commercial TV and radio advertising means they must always spend at least £4,000 on advertising every month. A graph of this cubic, between the values of $x = 4$ and $x = 17$, is shown below.



Some things to note from the graph:

- There are two *turning points* on the graph – one at about $x = 6$ and the other at about $x = 14$;

- One is known as a *local maximum*, the other is a *local minimum*;

- In this example, one of these points corresponds to maximum profit, the other corresponds to minimum profit;

- At both turning points, the gradient of the curve – and hence $\dfrac{dy}{dx}$ – is zero.

To find the exact x co–ordinate of each turning point, we can equate the derivate to zero and solve for x. For example, we know that

$$\begin{aligned}\frac{dy}{dx} &= 9.42 - 2 \times 1.17x + 3 \times 0.04x^2 \\ &= 9.42 - 2.34x + 0.12x^2.\end{aligned}$$

At each of the turning points shown in the graph, we know that the gradient is zero. Thus, we know that

$$9.42 - 2.34x + 0.12x^2 = 0,$$

and we know how to solve such a quadratic equation from the material in Chapter 1! Using the quadratic formula, we know that

$$x = \frac{-b \pm \sqrt{D}}{2a},$$

where D is the discriminant and is equal to $b^2 - 4ac$; a and b are the coefficients of the x^2 and x terms, respectively, and c is the constant.

In this example, $a = 0.12$, $b = -2.34$ and $c = 9.42$, giving

$$D = (-2.34)^2 - (4 \times 0.12 \times 9.42) = 0.954,$$

and so

$$x = \frac{2.34 \pm \sqrt{0.954}}{0.24},$$

giving

$$\begin{aligned}x &= \frac{2.34 + \sqrt{0.954}}{0.24} = 13.81971, \qquad \text{or} \\ x &= \frac{2.34 - \sqrt{0.954}}{0.24} = 5.680295.\end{aligned}$$

These are the *precise* values of x at which the curve has zero gradient – i.e. the advertising expenditure that will minimise/maximise profit – and this can be seen in the graph above.

So, spending £5,680 per month on advertising seems to be the optimal strategy, which will give monthly profits of

$$y = -7.36 + 9.42(5.680) - 1.17(5.680^2) + 0.04(5.680^3) = £15,729.$$

Spending £13,820 per month on advertising seems to give the worst outcome, with monthly profits of just

$$y = -7.36 + 9.42(13.820) - 1.17(13.820^2) + 0.04(13.820^3) = £4,944.$$

It is obvious from our graph which of the turning points gives the maximum profit and which gives the minimum profit. However, if we didn't have a graph of the function, the *second derivative test* can help us here. The second derivative of a function, sometimes written as

$$\frac{d^2y}{dx^2},$$

("dee 2 y by dee x squared") is just the derivative of the derivative. If x represents the x co–ordinate of our turning point, i.e. $\frac{dy}{dx}(x) = 0$, then:

- If $\frac{d^2y}{dx^2}(x) < 0$, our turning point is a maximum;
- If $\frac{d^2y}{dx^2}(x) > 0$, our turning point is a minimum.

In this example,

$$\frac{dy}{dx} = 9.42 - 2.34x + 0.12x^2.$$

Thus,

$$\frac{d^2y}{dx^2} = -2.34 + 0.24x.$$

At the first turning point, $x = 5.680295$, giving

$$\frac{d^2y}{dx^2} = -2.34 + 0.24 \times 5.680295 = -0.9767292,$$

which is negative, and so here we have a maximum turning point. At the second turning point $x = 13.81971$, giving

$$\frac{d^2y}{dx^2} = -2.34 + 0.24 \times 13.81971 = 0.9767304,$$

which is positive, and so here we have a minimum turning point.

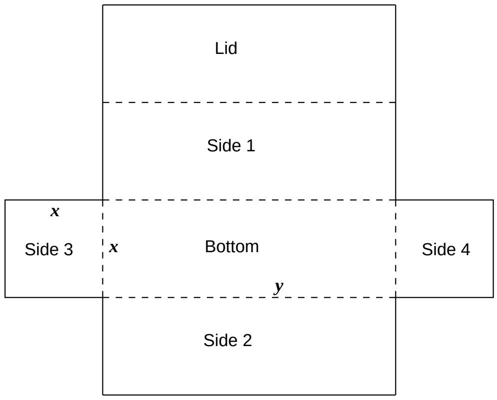
To summarise:

There are two turning points: one at $x = 5.680295$ and one at $x = 13.81971$. At $x = 5.680295$, the second derivative is negative and so we have a local maximum; at $x = 13.81971$ the second derivative is positive and so we have a local minimum. The local maximum/minimum correspond to maximum/minimum profit, and so spending £5,680 pounds on advertising is the optimal strategy, giving profits of just under £16,000. The worst strategy is to spend £13,820 on advertising, which will give a profit of just under £5,000.

Example 2.3

Choctastic! is a chocolate manufacturer. One of their gift boxes has 27 chocolates in it, requiring a volume of 1350 cubic centimetres. The company would like to work out the optimal dimensions of the required box to minimise the amount of packaging used (and hence minimise production costs).

The diagram below shows a "net" of the box that will be used; x and y are both in centimetres.



(a) Write down an expression for the volume of the box, in terms of x and y.

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(b) Formulate an expression for the surface area S of the box, in terms of x only.

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(c) Find $\frac{dS}{dx}$, and hence show that the optimal strategy is to use a cuboid box for the chocolates. What are the dimensions of this cuboid?

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(d) Use the second derivative test to show that you have minimised the amount of packaging used.

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2.3 Partial differentiation

As you will see in some of your other courses, most relationships in Economics, Accounting and Finance involve more than two variables. For example, the demand for a good depends not only on its own price but also on the price of substitutable or complementary goods, incomes of consumers, advertising expenditure and so on. Likewise, the output from a production process depends on a variety of inputs, including land, capital and labour. To analyse general economic behaviour we must extend the concept of differentiation to several variables.

Examples

1. If $f(x, y) = xy + 2y$, evaluate

(a) $f(3, 4)$;

(b) $f(4, 3)$.

2. If $g(x_1, x_2, x_3) = x_1^2 + x_2 - 3x_3$, evaluate

(a) $g(5, 6, 10)$;

(b) $g(0, 0, 0)$.

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Given a function of two variables, for example

$$z = f(x, y),$$

we can determine two first–order derivatives. The *partial derivative of f with respect to x* is written as

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x} \quad \text{or} \quad f_x,$$

and is found by differentiating f with respect to x, with y held constant. Similarly, the *partial derivative of f with respect to y* is written as

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y,$$

and is found by differentiating f with respect to y, with x held constant.

Examples

Find the first–order partial derivatives of the functions

(a) $f(x, y) = x^2 + y^3$;

(b) $f(x, y) = x^2 y$;

(c) $f(x, y) = x^2 y^3 - 10x$.

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In general, when we differentiate a function of two variables, the thing we end up with is itself a function of two variables. This suggests the possibility of differentiating a second time. In fact, there are four *second–order partial derivatives*. We write:

$$\frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 f}{\partial x^2} \quad \text{or} \quad f_{xx}$$

for the function obtained by differentiating twice with respect to x,

$$\frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad \frac{\partial^2 f}{\partial y^2} \quad \text{or} \quad f_{yy}$$

for the function obtained by differentiating twice with respect to y,

$$\frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{or} \quad f_{yx}$$

for the function obtained by differentiating first with respect to x and then with respect to y, and

$$\frac{\partial^2 z}{\partial x \partial y} \quad \text{or} \quad \frac{\partial^2 f}{\partial x \partial y} \quad \text{or} \quad f_{xy}$$

for the function obtained by differentiating first with respect to y and then with respect to x.

Examples

Find expressions for the second–order partial derivatives f_{xx}, f_{yy}, f_{yx} and f_{xy} for the functions

(a) $f(x, y) = x^2 + y^3$;

(b) $f(x, y) = x^2 y$.

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2.4 Chapter 2 practice questions

1. For each of the following, find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(a) $y = 6x^3$

(b) $y = 7x^4 - 3x^2$

(c) $y = \frac{1}{2}x^4 + \frac{1}{3}x^3 + 2x^2 + 6x - 3$

(d) $y = \frac{1}{3}x^3 + 5x^2 + 9x + 2$

(e) $y = 5\sqrt{x} - \pi x$

(f) $y = \frac{3}{x} + 2$

(g) $y = \frac{6}{\sqrt{x}} - \frac{1}{2}x^2$

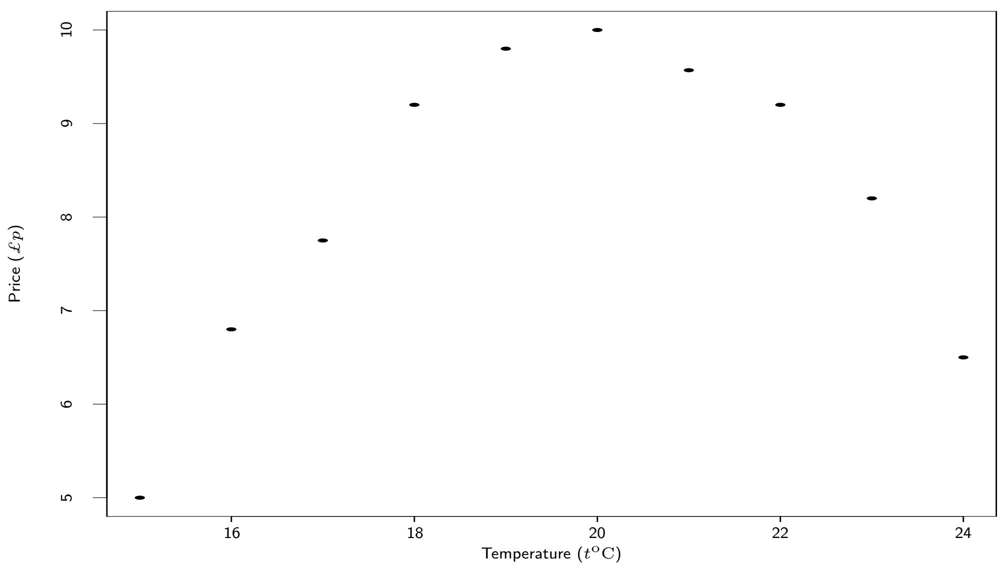
(h) $y = (x+7)(x-1)$

2. Find the co–ordinates of the turning point(s) of the functions in question 1 part (d), (e) and (h). Also, use the second derivative to find the nature of the turning point(s).

3. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the function in question 1(a) *from first principles.*

4. Recall Example 1.7 from the lecture notes:

The manager of a vineyard thinks there is a relationship between the average temperature during the grape–growing season (t°C) and the quality of the grapes produced and hence the selling price of resulting bottles of wine (£p). Over a ten year period the following relationship is revealed:



The quadratic model $p = -0.2t^2 + 8t - 70$ was used to explain how price varies with temperature. We also found that, to produce bottles of wine that will sell for at least £9, the critical range of temperatures is 17.8°C → 22.2°C. What temperature will *optimise* the selling price of the resulting bottles of wine? Use the second derivative test to verify that this is a maximum.

5. A company produces real fruit juice to be used in bottles of *Robinson's* diluted fruit juice concentrate. The company believes there is a relationship between their production rate (Q litres per hour) and their profit (£P per day). Specifically, they understand that

$$P = -Q^3 + 20Q^2 - 7Q - 1;$$

their maximum production rate is $Q = 16$ lites per hour.

Find the production rate that (i) minimises profit and (ii) maximises profit, verifying that you have a minimum/maximum turning point in each case. What is the maximum profit we can expect to achieve?

6. Find expressions for the second–order partial derivatives of the functions

(a) $f(x, y) = 5x^4 - y^2$;

(b) $f(x, y) = x^2y^3 - 10x$.

7. Find f_1, f_{11} and f_{21} in the case when

$$f(x_1, x_2, x_3) = x_1x_2 + x_1^5 - x_2^2x_3.$$