ACC1012 / ACC1053

Professional Skills / Introductory Quantitative Methods

Mathematics and Statistics

Semesters 1 & 2, 2017—2018

Lecturers: Dr. James Waldron & Dr. Kevin Wilson

School of Mathematics, Statistics & Physics

ACC1012/ACC1053

Professional Skills/Introductory Quantitative Methods

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Please see slides from the induction lecture for more detailed information on the course organisation, including information on computer based assessments and case studies

Classes

- There will be one *lecture* every week: Friday 9:30–10:30 in 1.03 in Newcastle Business School.
- Each student will attend **one** of 8 scheduled *workshops*:

Group	Day	Time	Place
A	Mon	12–1	Barbara Strang Teaching Centre 2.51
B	Mon	1–2	Barbara Strang Teaching Centre 2.51
C	Mon	4–5	Barbara Strang Teaching Centre 2.41B
D	Mon	5–6	Herschel Building TR1 (4th floor)
E	Tues	9–10	Barbara Strang Teaching Centre 3.31
F	Tues	12–1	Barbara Strang Teaching Centre 2.51
G	Tues	2–3	Barbara Strang Teaching Centre 2.51
H	Thurs	4–5	Barbara Strang Teaching Centre 3.31

Assessment

- **30%** Professional Skills (see other sessions)
- **70%** Mathematics & Statistics, consisting of:
  - Written Maths & Stats exam at the end of the academic year (50%)
  - Maths & Stats coursework, consisting of Computer Based Assessments (CBAs) and short written assignments based on case study material (20%)

The written exam will take place in May/June 2018; coursework will be set every few weeks, according to lecture material.

Semester 1 deadlines

Assessment	Deadline
CBA	Practice mode opens Thursday 5th October Assessed mode opens Thursday 12th October Deadline: 23:59 Wednesday 18th October
Case Study	Start work in tutorials w/b Monday 9th October Deadline: 4pm, Friday 3rd November
Case Study	Start work in tutorials w/b Monday 20th November Deadline: 4pm, Friday 12th January
CBA	Practice mode opens Thursday 30th November Assessed mode opens Thursday 7th December Deadline: 23:59 Wednesday 13th December

Deadlines for work in semester 2 will be announced in January

Other stuff

- Notes (with gaps) will be handed out in lectures – you should fill in the gaps during lectures.
- A summarised version of the notes will be used in lectures as slides.
- These notes and slides will be posted on the **course website** after each topic is finished, along with any other course material.
- The course website can be found via the "Additional teaching information" link on the Maths & Stats webpage, in the ACC1012/ACC1053 Blackboard page, or directly via:

  `http://www.mas.ncl.ac.uk/~nlf8/teaching/acc1012`

- Please check your University email account regularly, as course announcements will often be made via email.
- There is a course textbook available to buy at Blackwell's – not compulsory, but will be a good help!
- Calculators: the University has an approved list, available at

  `http://www.ncl.ac.uk/students/progress/exams/exams/CalculatorPolicy.htm`

Chapter 1

Linear and quadratic functions

Before we start the main topics of the Mathematics & Statistics element of the module, we will spend some time reviewing linear and quadratic functions. To some extent, you have already done this in the summer revision booklet. Nonetheless, it is important that you have completely mastered the basics before we move onto more challenging material; the aim of the first chapter, then, is to quickly cover some old ground, and we will do this through several examples.

1.1 Linear functions

Linear functions occur often in accounting and finance. For example,

$$y = 3x + 4$$

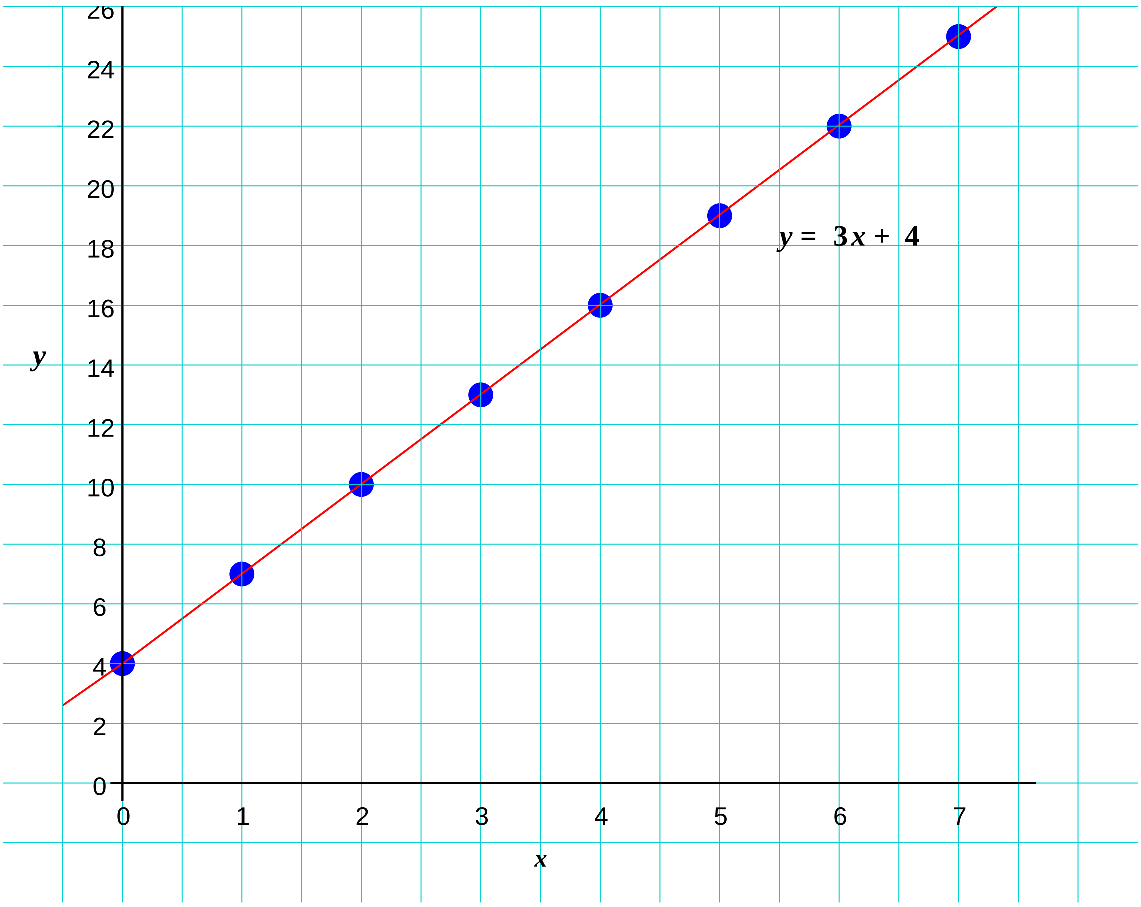
is a linear function. We know this is a linear function because when we produce a graph of the function, we get a straight line. For example, we can calculate the value of y for various values of x by substitution (see Section 1.1.5 of the summer revision booklet); for instance, when $x = 2$, we get

$$y = 3 \times 2 + 4 = 6 + 4 = 10,$$

bearing in mind the rules of BIDMAS (see Section 1.1.1 of the summer revision booklet). Similarly, for other values of x, we can obtain the following table of results:

x	0	1	2	3	4	5	6	7
y	4	7	10	13	16	19	22	25

Plotting x against y and joining up the points gives the graph shown overleaf. Notice that we get a straight line – and so we could have produced exactly the same graph using just two points instead of the eight we considered here (you only need two points to draw a straight line!).



Generally, we write down a linear function in the following way:

$$y = mx + c,$$

where m is the *gradient* and c is the (y)*–intercept*. Notice that, in a linear function, there are no "squared" terms or "cubed" terms in x (x^2 or x^3, respectively). In fact, the highest power of x is 1, since $x^1 = x$. Thus, we know a function is linear if the highest power in x is 1. If the highest power in x was 2, i.e. x^2, then we would have a *quadratic* function (see Section 1.2); the inclusion of a cubed term, i.e. x^3, would give a *cubic* function, and so on. Also notice that the gradient is always the number that is "stuck" to the x; that is, the gradient is the x *coefficient*. The intercept is the "other" number, and is sometimes called the *constant*. In our earlier example, where $y = 3x + 4$, the gradient is 3 and the intercept is 4 – you should be able to see the intercept on the graph. Some things to note about the gradient:

- The *sign* of the gradient (positive/negative) indicates the direction of the slope (uphill/downhill respectively)
- The *size* of the gradient indicates how steep the line is, where bigger numbers give steeper lines

We will come back to finding the equation of a line shortly.

1.1.1 Solving linear equations

In the summer revision booklet (Section 1.2.4), you should have thought about how to solve simple linear equations. Let's consider some more complicated examples. Don't forget to bear in mind the rules about dealing with negatives! (see Section 1.1.2 of the summer revision booklet).

Solve each of the following linear equations for x and t respectively.

1. $\dfrac{6}{x+4} = 4;$

2. $\dfrac{-4t+4}{5t-6} = -8.$

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1.1.2 Finding the equation of a line

See Sections 1.2.3 and 1.2.5 of the summer revision booklet for more details.

Suppose we are told that the gradient of a line is 6. We are also told that the line passes through the point $(3, 25)$. What is the equation of this line?

✎

Suppose we are told that a line passes through the points $(1, 3)$ and $(5, 35)$. What is the equation of the line?

✎

Example 1.1

The United Nations believes the annual consumption of rice in India (y kilograms per household) is a linear function of the unit cost (x US \$). They also know that, when the unit cost of rice is \$12, the annual consumption of rice is 40.8kg per household; when this unit cost doubles, the corresponding consumption decreases to 21.6kg per household.

(a) Using this information, obtain the linear function for demand in terms of cost.

(b) Find the unit cost of rice if the demand is 25kg per household.

1.1.3 Solving two linear equations simultaneously

In real–life accounting and finance problems, we often have more than one equation to solve. You might remember how to solve a pair of linear equations simultaneously from your GCSE maths course. If not, please see Section 1.2.6 of the summer revision booklet for more details, including how to solve a pair of simultaneous equations graphically.

Solve the following linear equations simultaneously; do *not* use a graph.

$$\begin{aligned} 4x + 8y &= 30 \\ 7x - 6y &= 5. \end{aligned}$$

✎

Example 1.2

A cookie company makes two types of biscuit: *standard* and *deluxe*. Suppose the company makes x batches of standard biscuits and y batches of deluxe biscuits every day.

(a) Each batch of standard biscuits requires 5kg of flour; each batch of deluxe biscuits requires 8kg of flour. Every day, exactly 98kg of flour must be used. Write down a linear equation in x and y for the total amount of flour used each day (in kg).

(b) Each batch of standard biscuits requires 2kg of butter; each batch of deluxe biscuits requires twice as much butter. Every day, exactly 44 kg of butter must be used. Write down a linear equation in x and y for the total amount of butter used each day (in kg).

(c) Solve your equations in parts (a) and (b) simultaneously to find out how many batches of standard and deluxe biscuits the company should make each day.

✎

1.1.4 Linear programming

Dynamic programming techniques were developed during the Second World War by a group of American mathematicians. They sought to produce mathematical models of situations in which all the requirements, constraints and objectives were expressed as algebraic equations. They then developed methods for obtaining the *optimal solution* – the maximum or minimum value of a required function.

In this section, we will attempt to formulate the requirements, constraints and objectives of real–life accounting and finance problems as linear equations (after all, this chapter is all about linear functions!); this branch of dynamic programming is referred to as *linear programming*, for obvious reasons. Thus, all algebraic expressions will be of the form

$$(\text{a number})x + (\text{a number})y;$$

for example,

$$\text{Profit} \quad = \quad 4x - 3y$$

is a linear equation for profit in terms of x and y, where x might represent our expenditure on advertising and y might be our costs.

Linear programming methods are some of the most widely used methods employed to solve management and economic problems. They have been applied in a variety of contexts, some of which will be discussed in this chapter, with enormous savings in money and resources.

The first step is to *formulate* a problem as a linear programming problem; the second step is to *solve* the problem. In fact, much of the work we do here will rely on our work on simultaneous equations in the previous section, as well as our earlier graph work; however, there will be an emphasis on being able to construct the equations in the first place (as we did in Example 1.1), not just solving the equations, and this might take a little time to master. In this section, we also need to think about linear *inequalities*; for example,

$$16x + 18y \quad \leq \quad 25$$

is a linear *inequality* in x and y. The role of inequalities will become apparent as we work through some examples.

Formulating linear programming problems

The first step in formulating a linear programming problem is to determine which quantities you need to know to solve the problem. These are called the *decision variables.*

The second step is to decide what the *constraints* are in the problem. For example, there may be a limit on resources or a maximum or minimum value a decision variable may take, or there could be a relationship between two decision variables.

The third step is to determine the objective to be achieved. This is the quantity to be maximised or minimised, that is, *optimised*. The function of the decision variables that is to be optimised is called the *objective function*.

The examples which follow illustrate the varied nature of problems that can be modelled by a linear programming model. We will not, at this stage, attempt to solve these problems but instead concentrate on producing the objective function and the constraints, writing these in terms of the decision variables.

Example 1.3

A manufacturer makes two kinds of chairs, A and B, each of which has to be processed in two departments, I and II. Chair A has to be processed in department I for 3 hours and in department II for 2 hours. Chair B has to be processed in department I for 3 hours and in department II for 4 hours.

The time available in department I in any given month is 120 hours, and the time available in department II, in the same month, is 150 hours.

Chair A has a selling price of £10 and chair B has a selling price of £12.

The manufacturer wishes to maximise his income. How many of each chair should be made in order to achieve this objective? You may assume that all chairs made can be sold.

At the minute, **we will not attempt to solve this problem**; we will simply *formulate* the situation as a linear programming problem. You'll notice that there's a lot of information given in the question – this is typical of a linear programming problem. Sometimes it's easier to summarise the information given in a table:

Type of chair	Time in dept. I (hours)	Time in dept. II (hours)	Selling price (£)
A	3	2	10
B	3	4	12
Total time available	120	150	

To formulate this linear programming problem, we consider the following three steps:

1. What are the *decision variables*? (i.e. which quantities do you need to know in order to solve the problem?)

2. What are the *constraints*?

3. What is the *objective*?

Step 1: Decision variables
To find out what the decision variables are, read through the question and identify the things you'd like to know in order to solve the problem. You can usually do this by going straight to the last sentence of the question. The last sentence here is

"How many of each chair should be made..."

Thus, we'd like to know the number of type A chairs to make, and the number of type B chairs to make. These are our decision variables, and are usually denoted with lower case letters (x and y if we have two decision variables, x, y and z if we have three, for example). Thus, our decision variables are

$$\begin{aligned} x &= \text{number of type A chairs made} \qquad \text{and} \\ y &= \text{number of type B chairs made.} \end{aligned}$$

Step 2: Constraints
Identifying the constraints is probably the hardest bit. To understand this bit, consider what could happen in each department. For example, if we focus on what could happen in department I:

Since:	the production of 1 type A chair uses 3 hours,
then:	the production of x type A chairs takes $3 \times x = 3x$ hours.
Similarly:	the production of 1 type B chair uses 3 hours,
so:	the production of y type B chairs takes $3 \times y = 3y$ hours.

The total time used is therefore

$$(3x + 3y) \text{ hours.}$$

Since only 120 hours are available in department I, one constraint is

$$\begin{aligned} (3x + 3y) \text{ hours} &\leq 120 \text{ hours}, \qquad \text{or just} \\ (3x + 3y) &\leq 120. \end{aligned}$$

Considering department II in a similar way, we get:

Since:	the production of 1 type A chair uses 2 hours,
then:	the production of x type A chairs takes $2 \times x = 2x$ hours.
Similarly:	the production of 1 type B chair uses 4 hours,
so:	the production of y type B chairs takes $4 \times y = 4y$ hours.

The total time used is therefore

$$(2x + 4y) \text{ hours.}$$

Since only 150 hours are available in department II, a second constraint is

$$\begin{aligned}(2x+4y) \text{ hours} &\leq 150 \text{ hours}, \qquad \text{or just}\\ (2x+4y) &\leq 150.\end{aligned}$$

In addition to the two constraints we have identified above, we also require that x and y are non–negative (since we can't make a negative number of chairs!), and so we also have the following two constraints:

$$\begin{aligned}x &\geq 0 \qquad \text{and}\\ y &\geq 0.\end{aligned}$$

These are called the *non–negativity constraints*.

Step 3: Objective function
Our objective here is to maximise income. If we make x type A chairs, then we get £$10 \times x =$ £$10x$, since each type A chair sells for £10.

Similarly, if we make y type B chairs, then we get £$12 \times y =$ £$12y$, since each type B chair sells for £12.

The total income is then

$$\pounds Z = \pounds(10x + 12y).$$

The aim is to maximise income, so we'd like to maximise

$$Z = 10x + 12y,$$

where z is the objective function.

Thus, to summarise, we have the following linear programming problem:

Maximise $Z = 10x + 12y$ subject to the constraints

$$\begin{aligned}3x+3y &\leq 120,\\ 2x+4y &\leq 150,\\ x &\geq 0 \qquad \text{and}\\ y &\geq 0.\end{aligned}$$

Example 1.4

Sportizus Clothing Company produce replica football shirts and replica rugby shirts for sale on the high street. Each shirt produced goes through a sewing process and a transfer process.

Each football shirt requires 8 minutes of sewing time and 9 minutes for the transfer process, whereas rugby shirts each require 5 minutes of sewing time and 15 minutes for the transfer process. In any given day, the total time available for the sewing process and transfer process is 10 hours and 15 hours respectively.

To meet current demand, *Sportizus* must produce at least 30 football shirts and 10 rugby shirts each day. The company sells football shirts and rugby shirts at a profit of £22 and £16 respectively.

How many of each type of shirt should *Sportizus* produce in order to maximise profits?

Let's start off with a table which summarises the question:

	Sewing time (mins)	Transfer process (mins)	Profit (P)
Football			
Rugby			
Total time			

Notice that the time available has been converted to minutes to be consistent with the other times given.

✎ **Step 1: Decision variables**

The decision variables are the number of football and rugby shirts to make. Let

x = and

y =

Step 2: Constraints

The constraints are:

sewing : and

transfer :

We do, of course, also have the non–negativity conditions; however, we are also told that we must make at least 30 football shirts and 10 rugby shirts to meet demand, giving:

$x \geq$ and

$y \geq$

Step 3: Objective function

Thus, to summarise, we have:

Solving linear programming problems

We can attempt to solve the two linear programming problems in Examples 1.3 and 1.4 graphically, using our earlier work on drawing graphs of linear functions. We do this by finding the *feasible region* for the problem, and then finding the point within this region which optimises our objective function.

Example 1.3 (revisited)

Recall that we had the following linear programming problem:

Maximise $Z = 10x + 12y$ subject to the following constraints:

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \quad \text{and} \\ y &\geq 0. \end{aligned}$$

To find the feasible region for this problem (i.e. the region on a graph which satisfies all of our inequalities), we proceed by indicating, on a diagram, the region for which all of the inequalities hold.

The first inequality is $3x + 3y \leq 120$. To show this on a graph, we first need to plot the line $3x + 3y = 120$.

- When $x = 0$, we have

$$\begin{aligned} 3 \times 0 + 3y &= 120 \quad \text{i.e.} \\ 3y &= 120 \quad \text{i.e.} \\ y &= 40. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 3x + 3 \times 0 &= 120 \quad \text{i.e.} \\ 3x &= 120 \quad \text{i.e.} \\ x &= 40. \end{aligned}$$

These points have been plotted on the graph overleaf and joined up to show the line with equation $3x + 3y = 120$. Since we want $3x + 3y \leq 120$, our region of interest lies on or below the line, and so we shade out the space above the line.

Now consider the second inequality $2x + 4y \le 150$. Again, to show this on a diagram, we first need to plot the line $2x + 4y = 150$.

- When $x = 0$, we have

$$\begin{aligned} 2 \times 0 + 4y &= 150 \quad \text{i.e.} \\ 4y &= 150 \quad \text{i.e.} \\ y &= 37.5. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 2x + 4 \times 0 &= 150 \quad \text{i.e.} \\ 2x &= 150 \quad \text{i.e.} \\ x &= 75. \end{aligned}$$

You should plot these points on the same graph overleaf, and hence draw the line with equation $2x + 4y = 150$. Since we want $2x + 4y \le 150$, our region of interest lies on or below the line, and so you should shade out the space above the line.

On the same graph we also shade out the inadmissible regions for the two non–negativity constraints.

The unshaded region in the graph shows the feasible region associated with our set of inequalities. What we must do now is find the point in that region which meets our objective – i.e. the point in that region which maximises income. One way of doing this is to also plot the objective function. Now our objective function is

$$Z = 10x + 12y,$$

where Z is our income. When Z takes different values we get a family of parallel straight lines. We need to choose a starting value for Z in order to be able to plot the objective function. It's often a good idea to try a value which is a multiple of both the coefficients of x and y. The coefficient of x is 10 and the coefficient of y is 12, so we could try a starting value of $Z = 120$. Thus, the objective function is now

$$10x + 12y = 120.$$

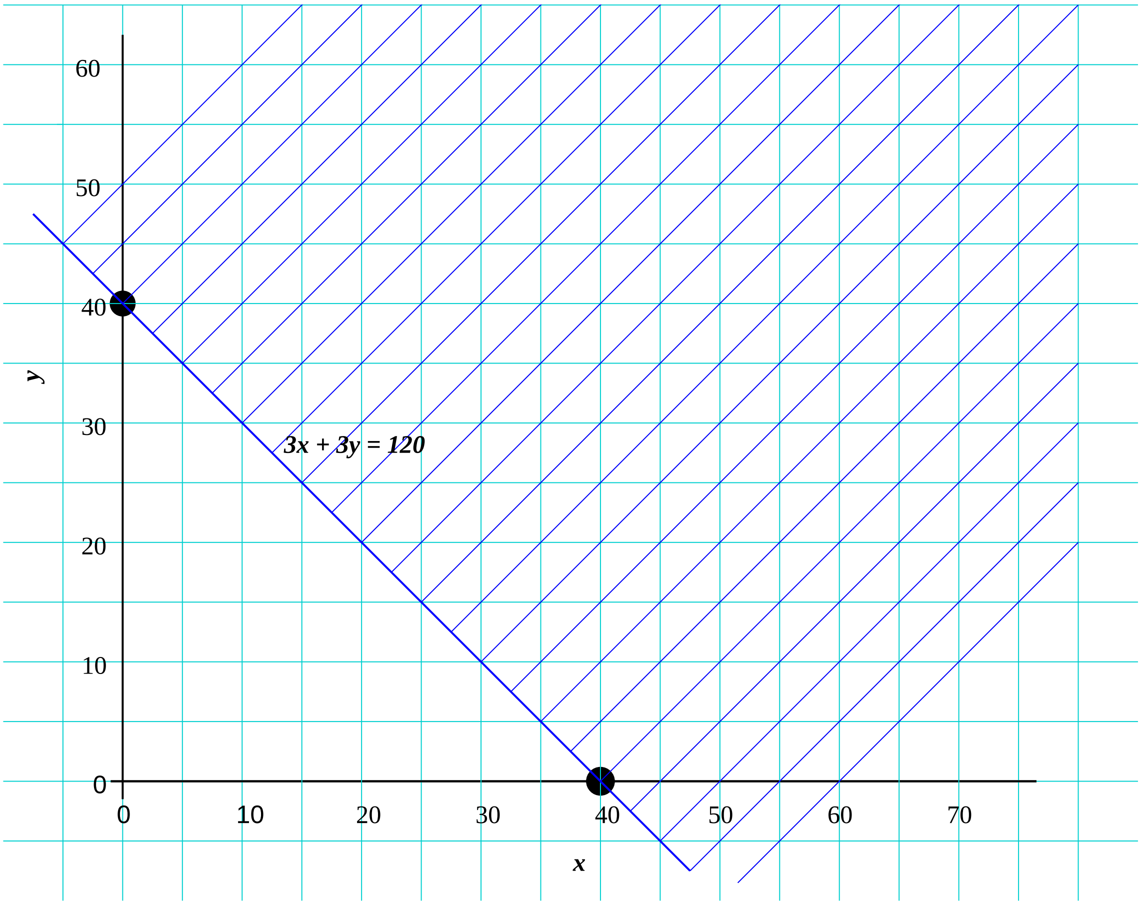
We can plot this line in the same way as before – i.e. consider what happens when x and y are zero.

- When $x = 0$, we have

$$\begin{aligned} 10 \times 0 + 12y &= 120 \quad \text{i.e.} \\ 12y &= 120 \quad \text{i.e.} \\ y &= 10. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 10x + 12 \times 0 &= 120 \quad \text{i.e.} \\ 10x &= 120 \quad \text{i.e.} \\ x &= 12. \end{aligned}$$



You should now plot this line on the graph as well. Notice that this line does not give the optimal income; the origin represents zero income, and we want to move as far away from this as possible. The largest value of Z (income) will occur at the point in the feasible region that is *furthest* from the origin, but still parallel to the objective line. You can find this point by sliding a ruler over the feasible region so that it is always parallel to the objective function drawn. This will enable you to identify the point that is furthest from the origin. In fact, you should notice that the point in the feasible region furthest from the origin (and parallel to the objective function) is the intersection of the two lines with equations $3x + 3y = 120$ and $2x + 4y = 150$.

All points in the feasible region satisfy our inequalities, but only one point maximises income. Once this point has been identified, we can simply "read off" the x and y values. Doing so give $\boldsymbol{x = 5}$ and $\boldsymbol{y = 35}$, and so, in order to maximise income, we should make **5 type A chairs** and **35 type B chairs**. This will give an income of

$$\begin{aligned} Z &= 10x + 12y \qquad \text{i.e.} \\ Z &= 10 \times 5 + 12 \times 35 \qquad \text{i.e.} \\ Z &= 50 + 420 \\ &= 470, \end{aligned}$$

i.e. £470.

Example 1.4 (revisited)

Recall that we had the following linear programming problem:

Maximise $P = 22x + 16y$ subject to the following constraints:

$$\begin{aligned} 8x + 5y &\leq 600, \\ 9x + 15y &\leq 900, \\ x &\geq 30 \qquad \text{and} \\ y &\geq 10. \end{aligned}$$

Solve this problem graphically to determine the number of football and rugby shirts *Sportizus* should make in order to maximise profit. Also, find the maximum achievable profit. Use the graph overleaf to help.

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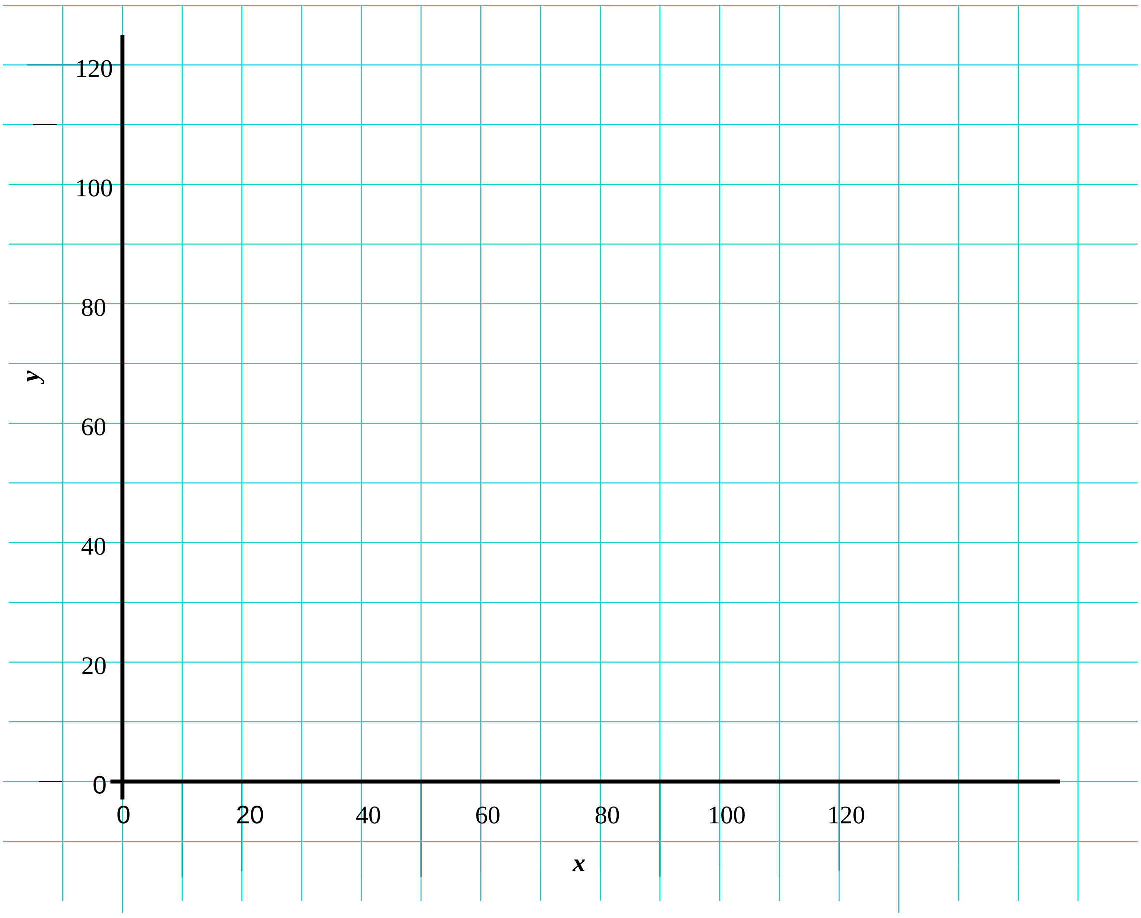


Both of these problems can also be solved algebraically by solving the pair of simultaneous equations shown in both graphs – you should try this yourself based on the material in Section 1.1.3.

1.2 Quadratic functions

As you might imagine, not everything in real–life can be represented by a straight line and, at best, the linear functions we looked at in Section 1.1 – such as linear cost functions and linear functions for profit – are simplifications. To this end, in this section we will consider the role of non–linear functions in the field of accounting and finance. In particular, we will think about *polynomial functions* where the power in x is greater than 1 (giving a non–linear graph). The simplest case here is the *quadratic function*, where the highest power of x in our polynomial is 2 – that is, at most we have an x^2 term. For example,

$$y = 5x^2 - 2x + 6$$

is a quadratic, since the highest power of x is 2; the polynomial

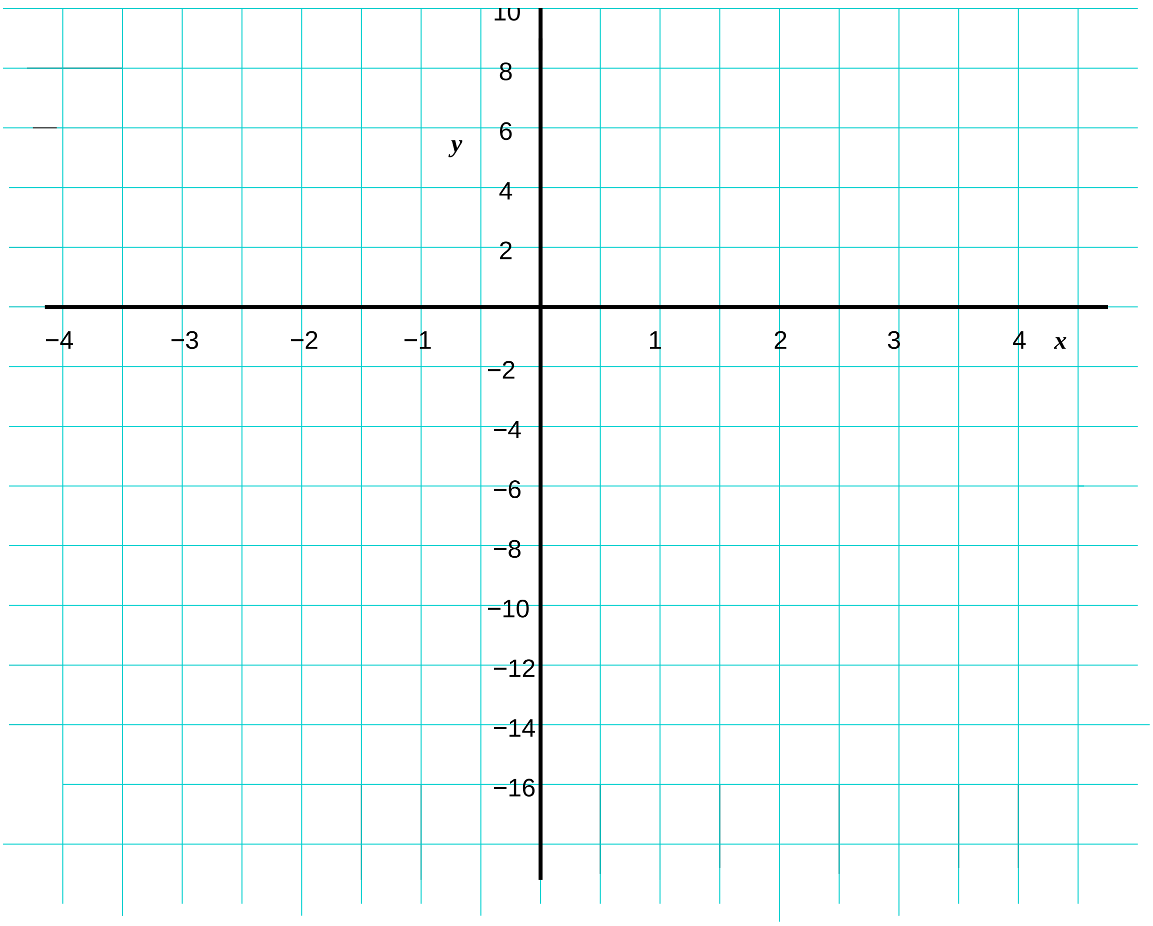
$$y = 2x^4 - 3x^2 + 9$$

is not a quadratic, since the highest power of x here is 4 (in fact, this is a *quartic*).

1.2.1 Plotting quadratic functions

In the space below, plot the graph of the function $y = x^2 - 2x - 15$.

✎



Example 1.5

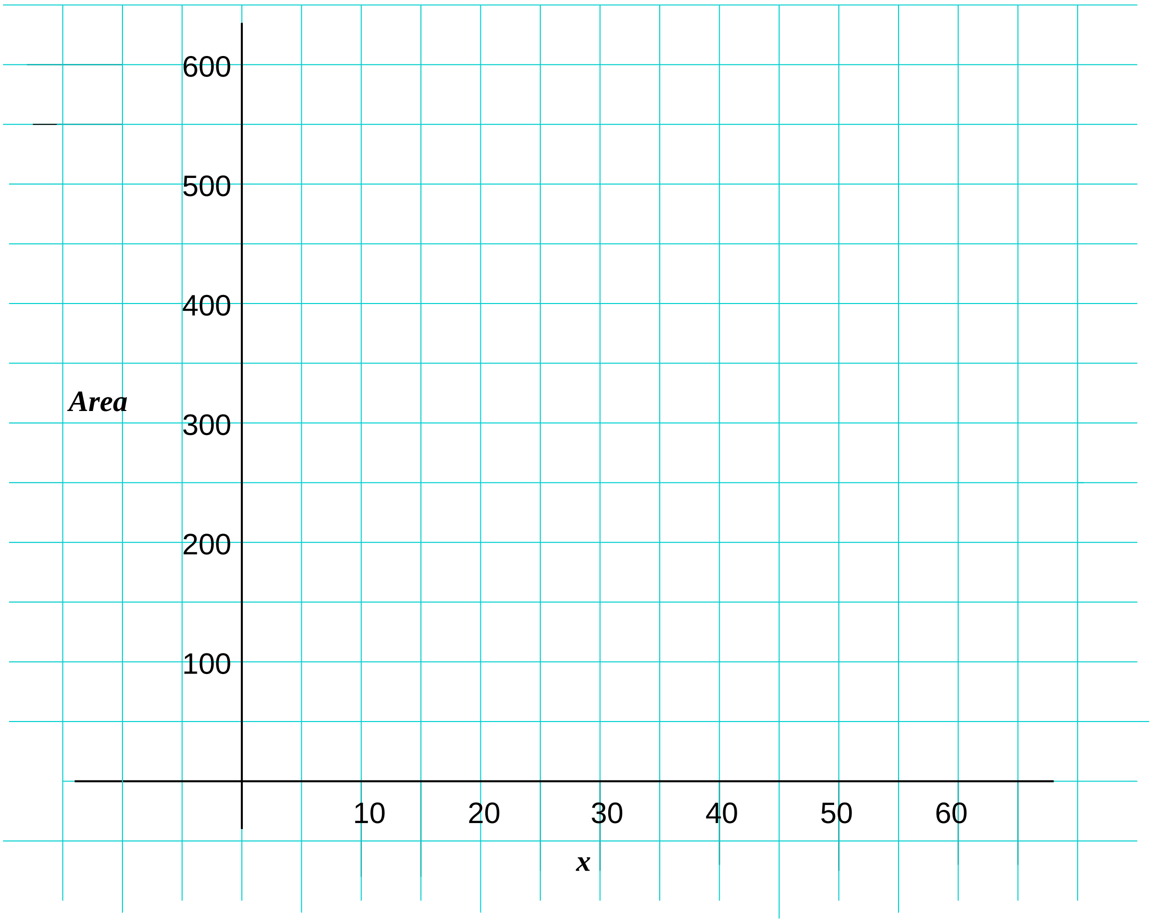
The estates manager of a zoo has 100 metres of fencing to construct a rectangular enclosure for some West African camels.

(a) Formulate a non–linear function for the area of the enclosure in terms of its width, x metres.

(b) Produce a plot of your non–linear function in (a) (see overleaf).

(c) Using your plot in (b), what length x would give the optimal area for the camel enclosure? What area would this give?

✎



As with linear functions, it would be useful to be able to plot or sketch a non–linear function without having to draw up a full table of results. For a quadratic function, it might be useful to know where the curve cuts into the x–axis, as well as the y–intercept of the curve. Consider the quadratic from the earlier example: $y = x^2 - 2x - 15$. We know the y–intercept occurs when $x = 0$, that is

$$y = 0^2 - 2 \times 0 - 15 = -15,$$

and this can be seen in our plot of this function. However, how can we tell, without drawing up a full table of results, where the curve will cut the x–axis? The curve cuts the x–axis when $y = 0$, that is when

$$x^2 - 2x - 15 = 0,$$

and so we would need to solve this equation for x. However, this is an example of a *quadratic equation*, and we haven't considered how to solve these yet...

1.2.2 Solving quadratic equations

We can visualise the solution(s) for x in the equation

$$ax^2 + bx + c = 0$$

by looking at a plot of the quadratic and noting where the curve intersects/touches the line $y = 0$, that is, the x–axis. However, the accuracy of our solutions obtained in this way will depend on the accuracy of our graph. We would also need to know what range of x–values to draw the graph over. Two approaches for solving quadratics a bit more mathematically will now be considered. In both, we will assume the general form of a quadratic as

$$y = ax^2 + bx + c;$$

the *discriminant*, D, is given by

$$D = b^2 - 4ac,$$

and has important properties in terms of how we classify the solutions to our quadratic.

Factorisation

If the discriminant is a perfect square – that is, if the square root of D as defined above is an integer value, then we can solve our quadratic by the method of factorisation. For more help with this, see Sections 1.3.1 and 1.3.2 of the summer revision booklet.

Consider the following quadratic equations:

1. $x^2 - 2x - 15 = 0$ (from earlier – see page 19)
2. $x^2 + 5x + 2 = 0$
3. $x^2 - 8x + 12 = 0$
4. $x^2 - 5x - 36 = 0$
5. $2x^2 - 7x + 5 = 0$

(a) For each, find the discriminant D.

✎

(b) Which equations can be solved by factorisation?

✎

(c) Solve the equations identified in (b) by factorisation.

✎

Quadratic formula

If we cannot factorise our quadratic, then we can use the quadratic formula in order to find the solution(s) to our quadratic equation. For example, equation (2) in the last question:

$$x^2 + 5x + 2 = 0$$

could not be factorised because the discriminant is

$$D = 5^2 - (4 \times 1 \times 2) = 25 - 8 = 17,$$

which is not a perfect square. However, all is not lost: the quadratic formula can help us here! This was developed by Indian mathematicians in around 628AD, although the formula as we know it today was not published until 1637 by a French mathematician. The solution(s) to the equation

$$ax^2 + bx + c = 0,$$

with discriminant $D = b^2 - 4ac$, are given by the formula

$$x = \frac{-b \pm \sqrt{D}}{2a}.$$

For example, consider the quadratic equation we plotted on page 19:

$$y = x^2 - 2x - 15.$$

The graph showed that when $y = 0$ the solutions were $x = 5$ and $x = -3$. Earlier in this section we also solved this equation by the method of factorisation. What about using the formula?

Note that $a = 1$, $b = -2$ and $c = -15$. Thus the discriminant is

$$D = (-2)^2 - 4 \times 1 \times -15 = 64.$$

So by the formula, we have

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{64}}{2 \times 1} \\ &= \frac{2 \pm 8}{2}. \end{aligned}$$

So the solutions are given by

$$x = \frac{2+8}{2} = \frac{10}{2} = 5,$$

or

$$x = \frac{2-8}{2} = \frac{-6}{3} = -3,$$

exactly the solutions we obtained graphically and by factorisation!

Solve the quadratic equation

$$x^2 + 5x + 2 = 0.$$

✎

Example 1.6

A company models its annual profits using the function

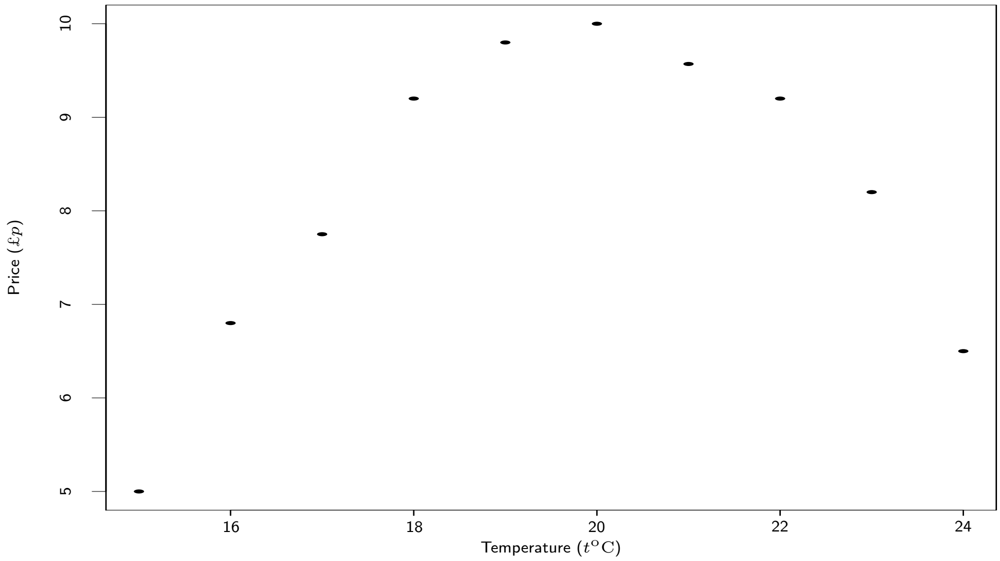
$$P(x) = x^2 + 20x - 300,$$

where P represents profits and x is the number of units sold. In 2011 their profits were £167,700. How many units of their product did they sell?

✎

Example 1.7

The manager of a vineyard thinks there is a relationship between the average temperature during the grape–growing season (t^oC) and the quality of the grapes produced and hence the selling price of resulting bottles of wine (p pounds). Over a ten year period the following relationship is revealed:



(a) Which model would best suit the relationship shown?

* $p = 0.2t + 4.4$
* $p = 0.2t^2 - 8t + 70$
* $p = -0.2t^2 + 8t - 70$

(b) Suppose the manager needs to sell this variety of wine in 2013 for at least £9 per bottle. What is the critical range of temperatures required to meet this price?

✎

Summary

Earlier, we discussed that the discriminant $D = b^2 - 4ac$ has important properties in terms of how we classify he solutions to our quadratic equation. In summary:

- If D is a perfect square, then we can solve the quadratic by factorisation; otherwise, we can use the formula (in fact, the formulae will always work!)
- If $D > 0$, our quadratic will have *two real solutions* – that is, the curve will cut the x–axis in two places
- If $D = 0$, then our quadratic will have *repeated solutions* – that is, our graph only just touches the x–axis (in effect giving just a single solution)
- If $D < 0$, then we cannot operate the quadratic formula as we cannot take the square root of a negative – at least, not in the real number system. Actually, such a quadratic has *complex roots* making use of the complex number system, but this goes beyond the scope of this module

1.3 Higher–order polynomials

In more complicated accounting and finance problems, linear and/or quadratic functions might still not be adequate enough to model our situation realistically. Higher–order polynomials can be used, such as cubics:

$$y = ax^3 + bx^2 + cx + d,$$

or even quartics:

$$y = ax^4 + bx^3 + cx^2 + dx + e;$$

or even quintics! –

$$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f.$$

We will return to such functions in Chapter 2 of this course.

1.4 Chapter 1 practice questions

1. What are the intercept and gradient of the following linear functions?

(a) $y = 2x + 5$

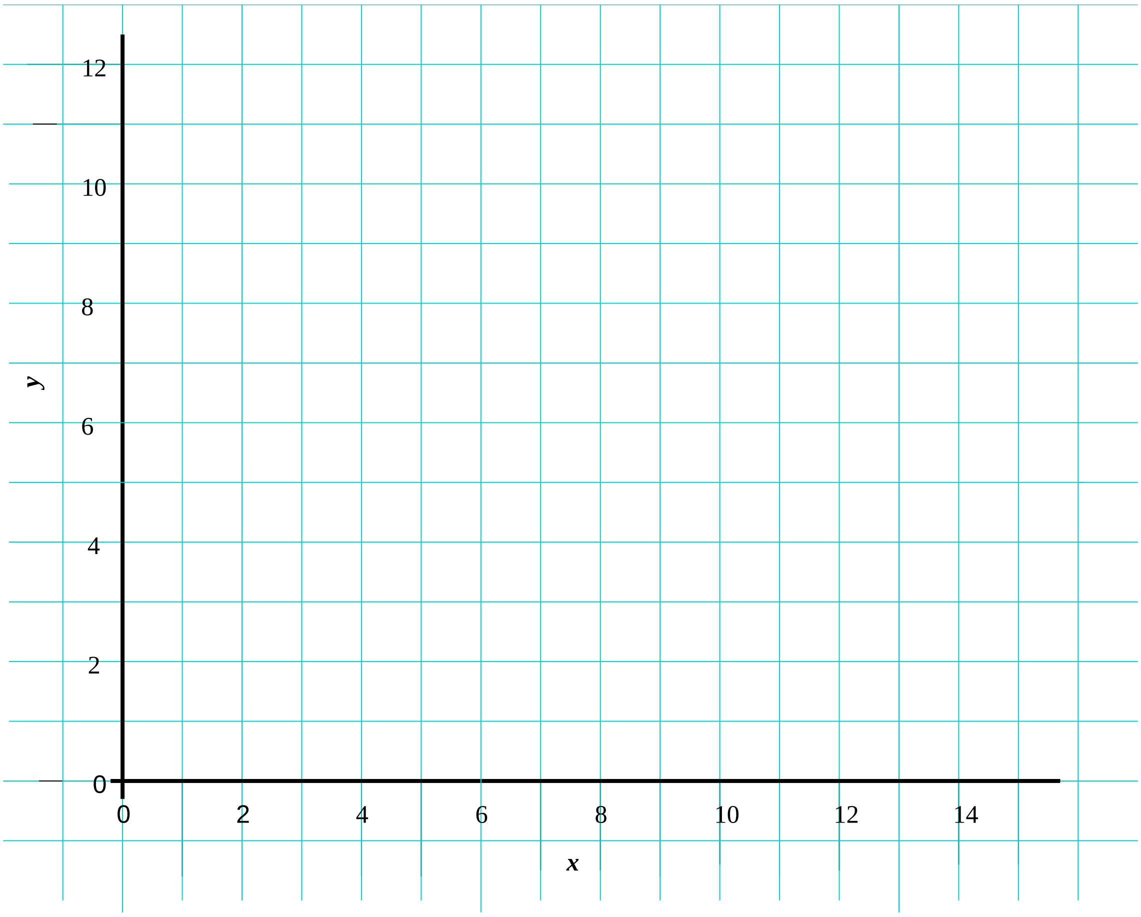
(b) $y = 12 - 2x$

(c) $8y = 2x + 16$

(d) $y = x$

2. Put the linear functions in question 1 in order of "steepness", from most steep to least steep.

3. On the graph below, plot each of the linear functions given in question 1.



4. A recent British Gas investigation examined factors which influenced the length of time it takes customers to pay their utility bills (see the British Gas website for details about this investigation and a fully downloadable report). Their report revealed that the size of a customer's bill (£x) was the most important factor influencing the time it took customers to pay this bill (y, in days).

In particular, British Gas express the time it takes a customer to pay their bill as a linear function of the size of their bill:

$$y = 0.06x + 7.44.$$

(a) What are the intercept and gradient of this linear function?

(b) If Kristina Bell's gas bill is £450, how long can we expect it will take her to pay this bill?

(c) Suppose it took Maeve Campbell 10 days to pay her gas bill. Solve the above linear equation for x to find out how Maeve's gas bill was.

5. Jake Bennett needs a new carpet for his bedroom. His room is rectangular in shape, and the length of the room is 3 times the width. Let x be the width of the room, in metres.

(a) Write down an expression for the perimeter of Jake's bedroom. Simplify this expression as far as possible.

(b) Write down an expression for the area of Jake's bedroom.

(c) Which one of your expressions in (a) and (b) is *linear*?

6. Alexander Booth is an entrepreneur who has recently developed a miracle hangover cure. The number of hours until a hangover is cured (y) is thought to be related to the dose given (x mg).

Specifically, if a person suffering with a hangover is not given any of the drug at all, the hangover will pass within 10 hours. However, a 50mg dose should see the hangover pass within 2 hours.

(a) Assuming y is a linear function of x, find the intercept c and the gradient m, and hence write down the linear function.

(b) Assuming side effects are negligible regardless of the dose, what is the optimal dose to give hangover sufferers?

7. Using a mobile phone costs £30 per month, and an additional 16 pence per minute of use (texts are free).

(a) Write down a linear function of cost in terms of minutes used each month. Make sure you clearly define any variables.

(b) Suppose you make 3.5 hours worth of calls this month. How much should your bill be?

(c) You now upgrade your mobile phone. Under your new tariff you pay £25 per month and only 5 pence per minute of use. Text messages are still free, but your new phone is a smart phone and you will be charged 50p per gigabyte of data you use. Write down a linear function for cost in terms of call time and data, clearly defining any variables.

8. *Tallentire TV* are a company producing parts for plasma televisions. They currently have a contract to produce a certain part for a batch of Samsung's latest 3D televisions.

The graph overleaf shows their *linear cost function* – that is, their total costs (£y hundred) as a function of the number of parts they need to make (x).

(a) What is the y–intercept?

(b) Calculate the gradient.

(c) Use your answers to parts (a) and (b) to write down the linear cost function.

(d) Give a practical interpretation of the intercept and gradient.

(e) Use the graph, and your linear function, to find total costs given that Samsung require parts for a batch of 24 televisions.

9. Which of the following are *linear* equations in x?

(a) $5x + 7 = 27$

(b) $3x - 9 = -21$

(c) $7(2x - 4) = 154$

(d) $3x(4 - 2x) = -90$

(e) $\dfrac{3x - 2}{-2(4x - 1)} = -\dfrac{8}{23}$

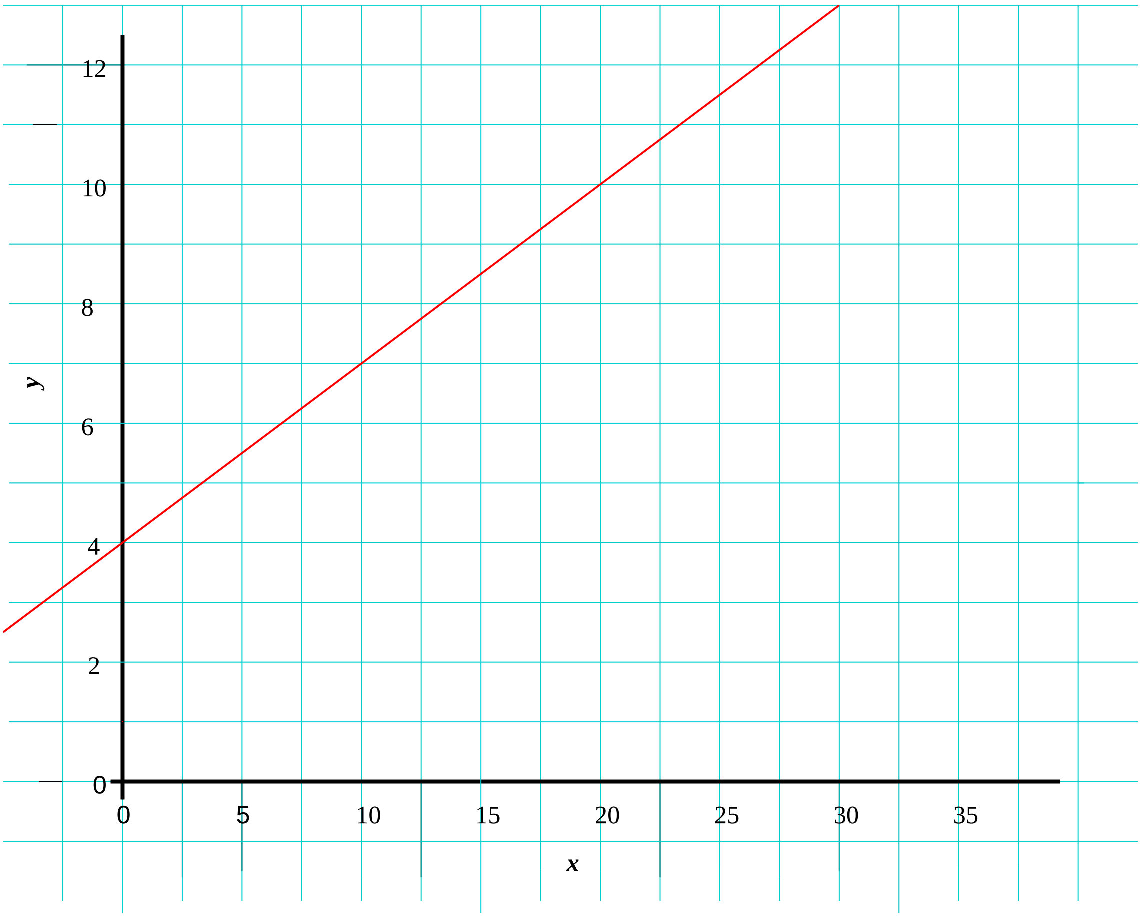
(f) $\dfrac{10x^2 + 30}{5x} = 2x + 3$

10. Solve the linear equations in question 9 for x.

11. Andreas Doulappas has secured an internship at a leading accountancy firm. He receives double pay for every hour he works over and above 38 hours per week. Last week, he worked 48 hours and earned £812.

(a) Formulate a linear equation for this scenario in terms of x, Andreas' regular hourly wage.

(b) Solve your equation in (a) to find Andreas' regular hourly wage.



12. *Wong Enterprises* have a team of quantitative analysts who forecast the profit margins of their clients. For one company, they show that profit is directly related to total income and overheads, that is

$$\text{Profit} = \text{total income} - \text{overheads},$$

where total income itself is a linear function of advertising expenditure (£x):

$$\text{Total income} = 6(x - 2000).$$

If the company's profit last month was £188,000, and their overheads were £10,000, how much did they spend on advertising?

13. (a) The gradient of a straight line is -0.75. If the line passes through the point $(-6, 1)$, find the equation of the line in the form $y = mx + c$.

(b) Another straight line passes through the points $(2, 11)$ and $(5, 38)$. Find the equation of this line in the form $y = mx + c$.

14. *Product placement*, sometimes known as *embedded marketing*, is a form of advertising where branded goods or products are placed in a context usually devoid of advertisements – such as movies, music videos or TV shows.

Researchers at *Samsung* believe there could be a linear relationship between the time at which a product appears in a movie and the proportion p of people who can recall the brand at the end of the movie. In particular, they propose the following linear model:

$$p = -0.06 + 0.01t,$$

where t is the time (in minutes), from the start of the film, at which the product placement occurred.

(a) In terms of the gradient, explain the relationship between p and t.

(b) After an advance screening of the James Bond film *Skyfall*, 105 of the 120 viewers in the auditorium were able to tell researchers that a *Samsung* mobile phone was featured during the film. According to the linear model above, after how many minutes did the product placement occur?

(c) Explain why, *in general*, such a model could be flawed.

15. *Ewart Electricals* produce circuitboards for televisions and Blu–Ray players for leading brands such as *Hitachi* and *Panasonic*. The company use linear cost functions for the costs they incur; that is, overall costs (y) are a multiple of the number of circuitboards produced (x), minus any fixed costs.

(a) For a batch of televisions, each circuitboard cost the company £20 to produce. For a particular order, the company must make 50 circuitboards at a total cost of £1,150. Find the company's linear cost function for televisions, and identify the fixed costs here.

(b) For one batch of Blu–Ray players, the company must make 120 circuitboards giving a total cost of £3,040. Another batch of players requires 175 circuitboards and incurs total costs of £4,250. Find the company's linear cost function for Blu–Ray players, and identify the fixed costs and cost per circuitboard here.

16. Solve the following pairs of linear equations simultaneously for (x, y), (p, q) or (A, B):

$$\begin{aligned} 2x + 3y &= 18 \\ 2x + y &= 14 \end{aligned}$$

$$\begin{aligned} 4x + 5y &= 65 \\ 3x - 3y &= -12 \end{aligned}$$

$$\begin{aligned} 6p + 3q &= 108 \\ 4p + 7q &= 122 \end{aligned}$$

$$\begin{aligned} 6A + 2B &= 20 + 5B \\ 5A + 10B + 6 &= 13B + 17 \end{aligned}$$

17. The *Monster Party Company* produce two types of party pack. Their "Ghastly" party pack contains 10 balloons and 64 sweets. Their "Devilish" party pack contains 20 balloons and 16 sweets. Each week, the company must use 3000 balloons and 8000 sweets in their party packs.

(a) How many "Ghastly" and "Devilish" party packs can the company make each week? Solve this problem graphically *and* algebraically.

(b) The company sells each "Ghastly" party pack at a profit of £1.20 and each "Devilish" party pack at a profit of £1.80. How much profit can they expect to make each week?

18. Kuddly Pals Co. Ltd make two types of giant soft toy: bears and cats. The quantity of material needed and the time taken to make each type of toy is given in the table below.

Toy	Material (m^2)	Time (minutes)
Bear	5	12
Cat	8	8

Each day the company can process up to $2000m^2$ of material and there are 48 worker hours available to assemble the toys.

The profit made on each bear is $\pounds 1.50$ and on each cat is $\pounds 1.75$. Kuddly Pals Co. Ltd wishes to maximise its daily profit.

(a) Formulate the company's situation as a linear programming problem.

(b) Draw a suitable diagram to enable the problem to be solved graphically, indicating the feasible region and the direction of the objective line.

(c) Use your diagram to find the company's maximum profit, $\pounds P$.

(d) Verify your solution algebraically.

19. A chocolate manufacturer produces two types of chocolate bar: *Asteroids* and *Blackholes*. Production of an Asteroid bar uses 10g of cocoa and 1 minute of machine time, whereas a Blackhole bar requires 5g of cocoa and 4 minutes of machine time. Altogether, 2kg of cocoa are available each day. The company employs a single machine operator who works an eight hour day.

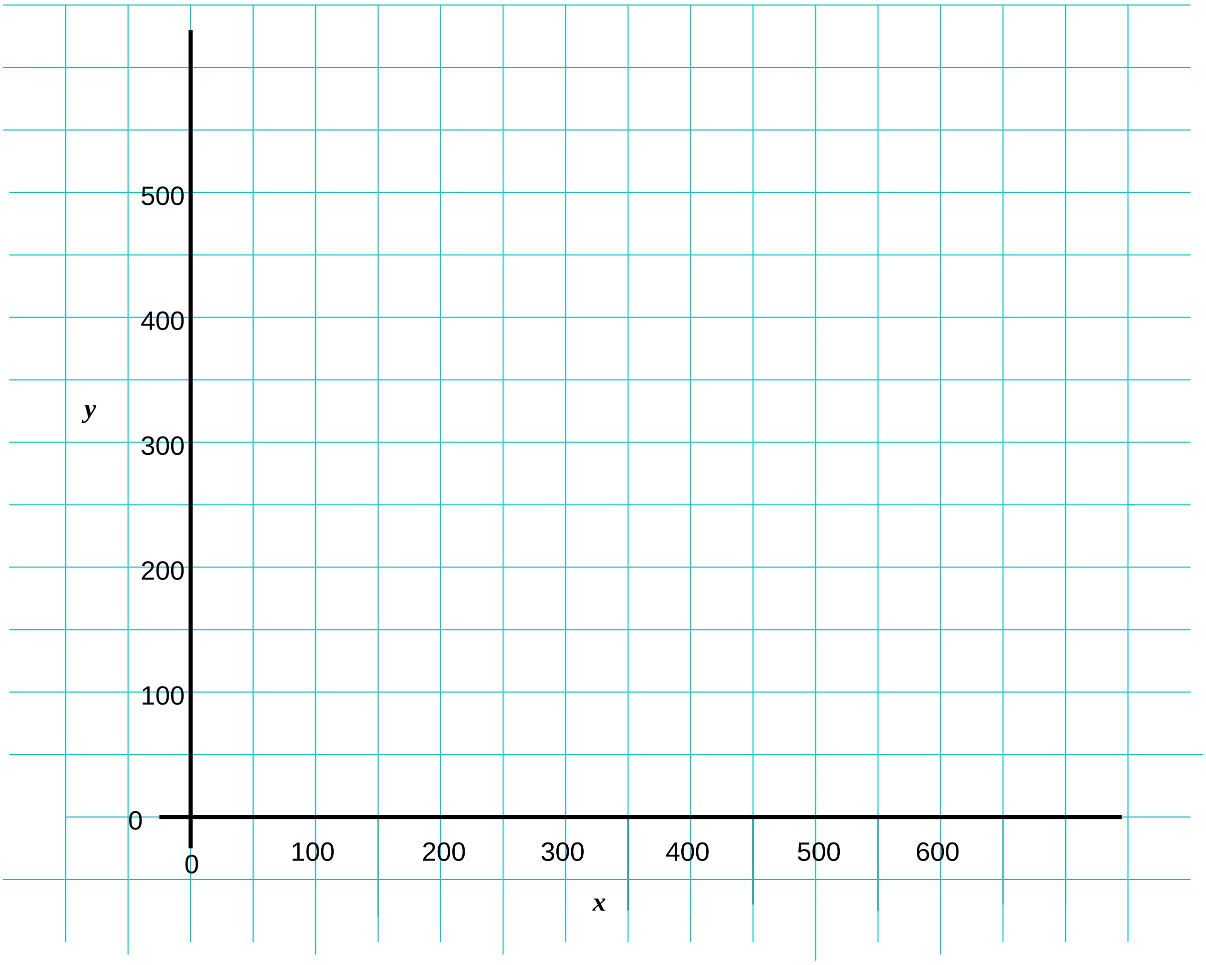
The manufacturer must make at least 50 Asteroid and 50 Blackholes each day to keep up with demand. The manufacturer makes 10p profit from each Asteroid bar and 20p profit from each Blackhole bar.

(a) Formulate the chocolate manufacturer's situation as a linear programming problem.

(b) Draw a suitable diagram to enable the problem to be solved graphically, indicating the feasible region and the direction of the objective line (use the graph below).

(c) Use your diagram to find the company's minimum and maximum profit, £P.

(d) Now solve this problem algebraically to verify your solution to part (c).



20. Which of the following are *quadratic* equations in x?

(i) $5x + 9 = 2x - 3$

(ii) $x^2 + 11x + 28 = 0$

(iii) $x^2 - 4x = 5$

(iv) $x^2 + 3x + 1 = 0$

(v) $\dfrac{6x^2 + 5x + 26}{2x} = 3x + 4$

(vi) $x^2 + 10 = 6x$

(vii) $x^2 - 16x + 64 = 0$

21. For each of the quadratic equations in question 20, calculate the discriminant. Using this value,

(a) decide whether the quadratic will have two distinct solutions, repeated solutions or complex solutions;

(b) decide which could be solved using the method of factorisation.

22. (a) Solve the quadratic equations you have identified in question 21(b) using the method of factorisation.

(b) For any remaining quadratics in question 20 that can be solved, use the quadratic formula to solve for x.

23. The amount of profit a company makes, £π million, is related to their advertising expenditure £q million according to the following function:

$$\pi(q) = -2q^2 + 9q - 4.$$

Find the company's break–even points, that is, their advertising expenditure that gives zero profit. Also, produce a sketch of $\pi(q)$.

24. In the 1950's, the Japanese statistician Genichi Taguchi (1924–2012) developed a technology for applying mathematics and statistics to improve the quality of manufactured goods. *Taguchi quality loss functions* are now used frequently in the fields of accounting, finance, business and economics. In general, the Taguchi loss function is given by

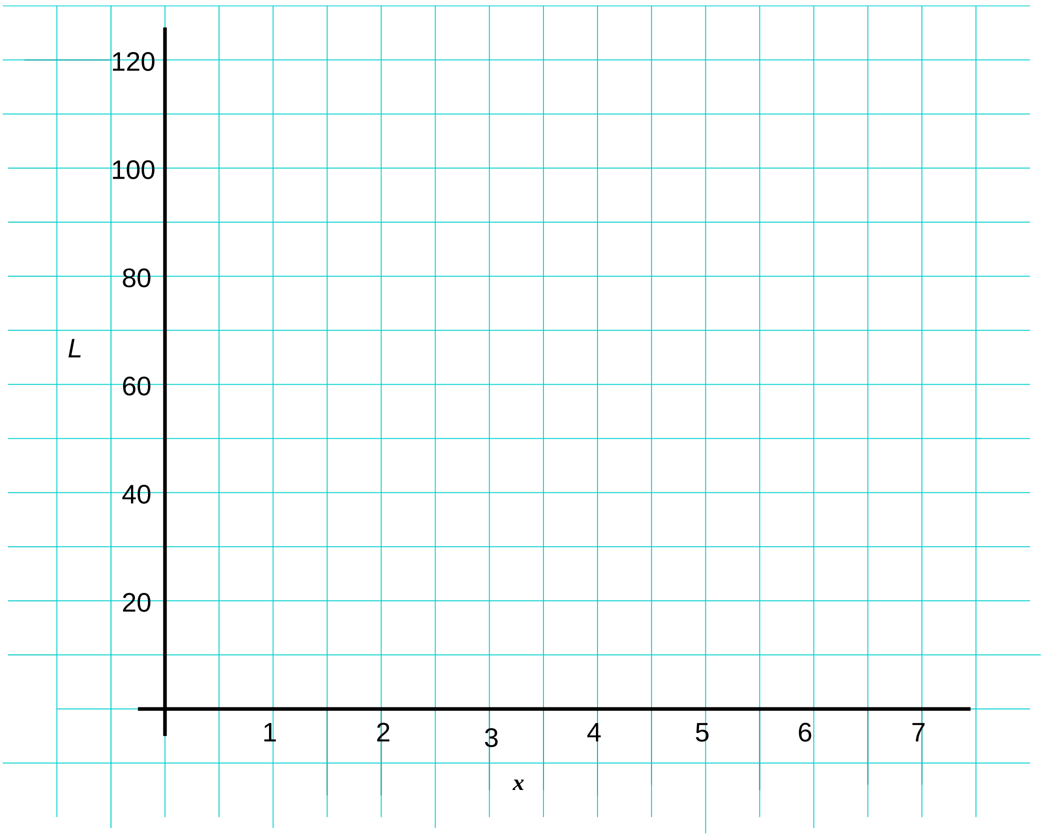
$$L = k(x - m)^2,$$

where

- L is the monetary loss
- x is the quality characteristic (e.g. concentration, diameter,...)
- m is the target value for x
- k is a constant

Peng Power produce printed circuitboards for the new iPhone 5. Employees use a machine to precision–drill tiny holes, of diameter 2mm, into the circuitboards. Mistakes can be rectified, but at a cost to the company. In fact, a Taguchi loss function can be used to estimate the monetary loss, L pence, for each circuitboard that is drilled erroneously, where $k = 4$.

(a) Write down the company's Taguchi loss function, and plot this as a smooth curve on the graph below (you might want to draw up a table of results).



(b) Look at your graph in part (a).

(i) What value of x gives zero loss?

(ii) Why does your answer to part (i) make sense intuitively?

(c) Verify your answer to part (b)(i) by expanding the company's Taguchi Loss function and solving for $L = 0$ using the quadratic formula.

(d) What is the value of the *discriminant*? What does this mean in terms of the solutions for this quadratic? Can you see this in your graph in part (a)?