Chapter 5

Continuous probability models

What we'll cover...

- **Probability density functions**
- The **Normal distribution**
- The **Uniform distribution**
- The **exponential distribution**
- **Poisson processes**

5. Continuous probability models

We have seen how **discrete** random variables can be modelled by discrete probability distributions such as the **binomial** and **Poisson** distributions.

We now consider how to model **continuous** random variables.

5.1 Introduction

A variable is **discrete** if it takes a **countable** number of values.

For example,

- the number of **blue** cars that I count in a 5 minute period
- the number of **heads** observed when I flip a coin ten times
- Shoe sizes: $1, \ldots, 12, 13, 1, 2, \ldots$
- $r = 0, 0.1, 0.2, \ldots, 0.9, 1.0$

In contrast, the values which a **continuous variable** can take form a **continuous scale**, with no “jumps”.

For example,

- Height
- Weight
- Temperature

5.1 Introduction

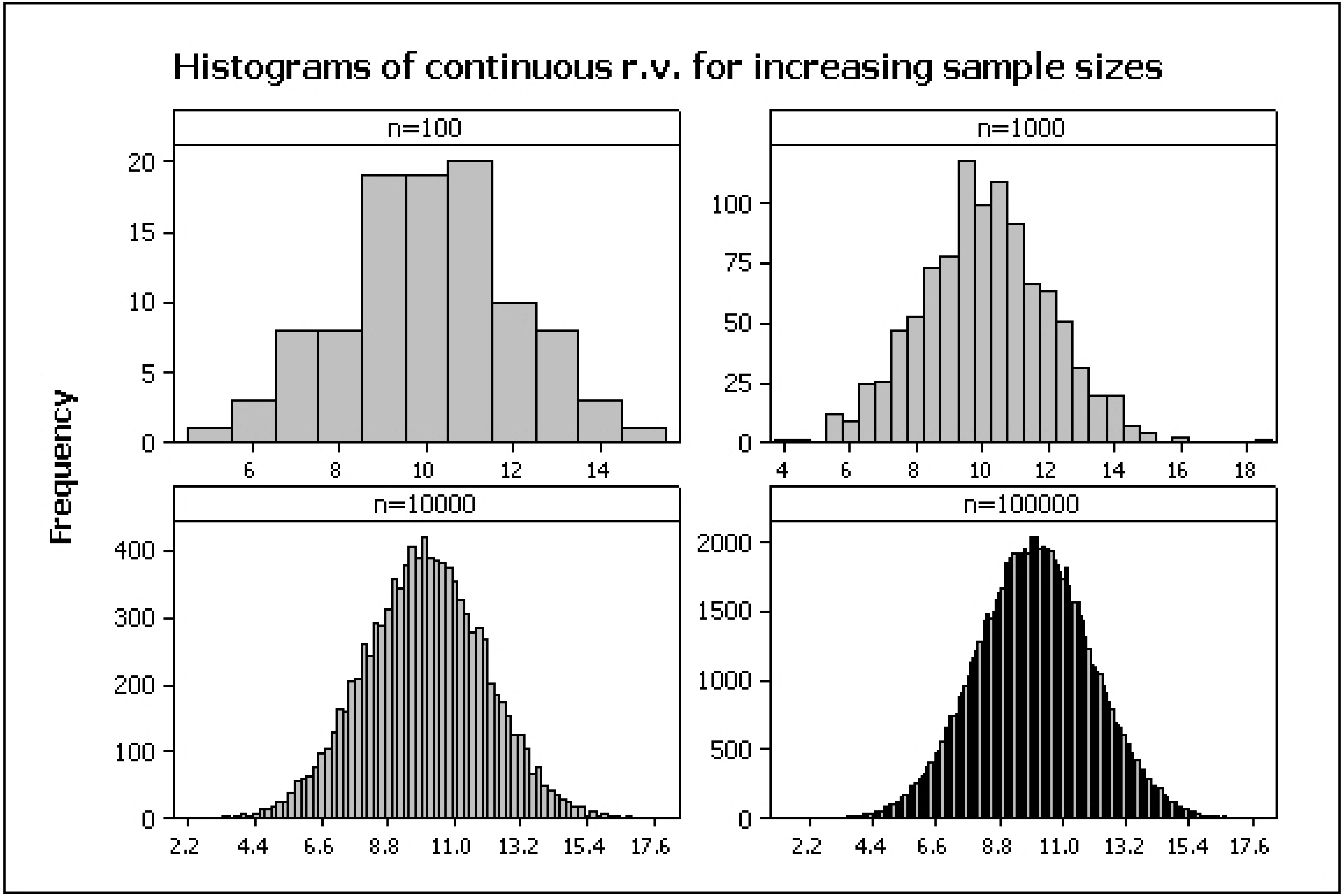
Think about **height**.

- In practice, we might only record height to the **nearest cm**
- If we could measure height *exactly* we'd find that everyone had a **different height**
- This is the essential difference between discrete and continuous variables
- If there are n people on the planet, the probability that someone's height is x would be $\frac{1}{n}$
- As n gets bigger and bigger, this probability tends to zero!!

5.1 Introduction

Consider taking a sample of values from the continuous random variable ***X***. This is what we'd observe as our sample got bigger and bigger:

5.1 Introduction



5.1 Introduction

- As the sample size gets bigger, the **interval widths get smaller**
- the jagged profile of the histogram **smooths out** to become a curve
- When the sample size is infinitely large, this curve is known as the **probability density function** (pdf)

Features of the probability density function

The key features of pdfs are:

1. pdfs never take negative values
2. the area under a pdf is one: $P(-\infty < X < \infty) = 1$
3. areas under the curve correspond to probabilities
4. $P(X \leq x) = P(X < x)$ since $P(X = x) = 0$.

5.1 Introduction

Over the next two weeks we will consider some particular probability distributions that are often used to describe continuous random variables.

We start with the most **important**, most **widely–used** statistical distribution of all time...

...wait for it...

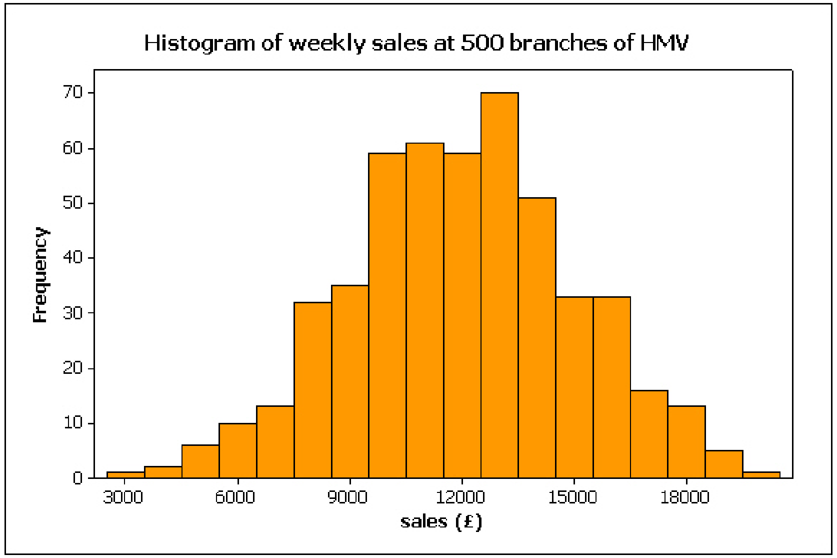
The Normal Distribution

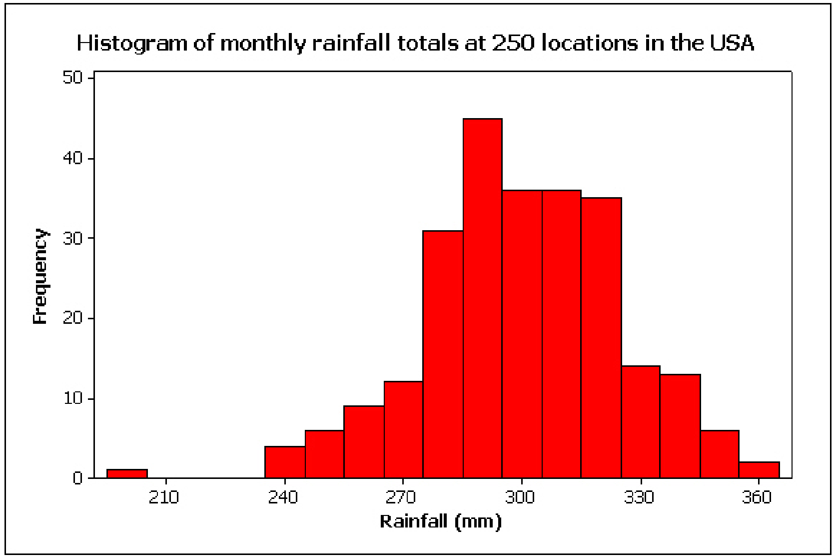
5.2.1 Introduction

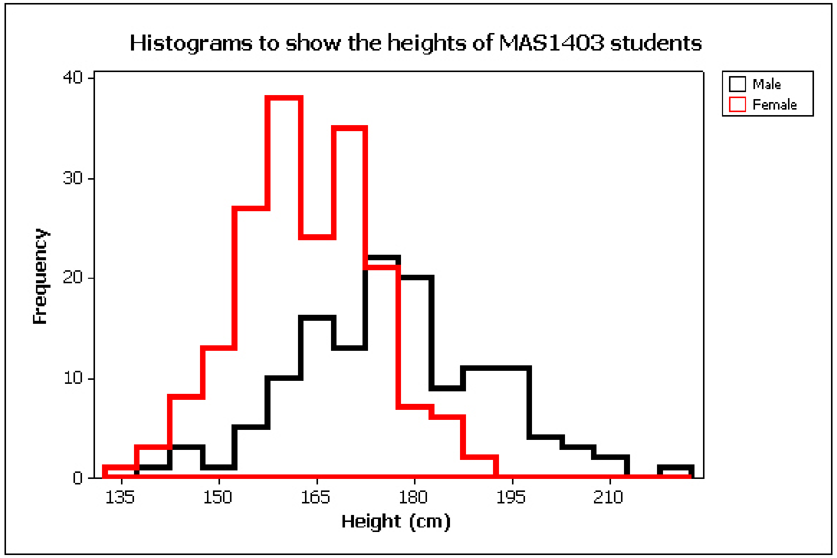
The **Normal distribution** is without doubt the most widely–used statistical distribution in many practical applications:

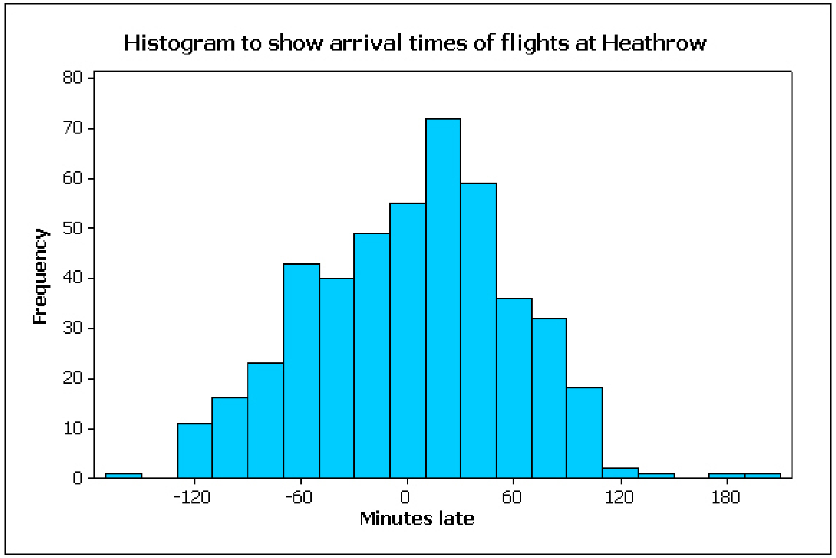
- Normality arises **naturally** in many physical, biological and social measurement situations
- Normality is important in **Statistical inference**
- The normal distribution has many guises:
  - Gaussian distribution
  - Laplacean distribution
  - “bell–shaped curve”

Some real–life examples









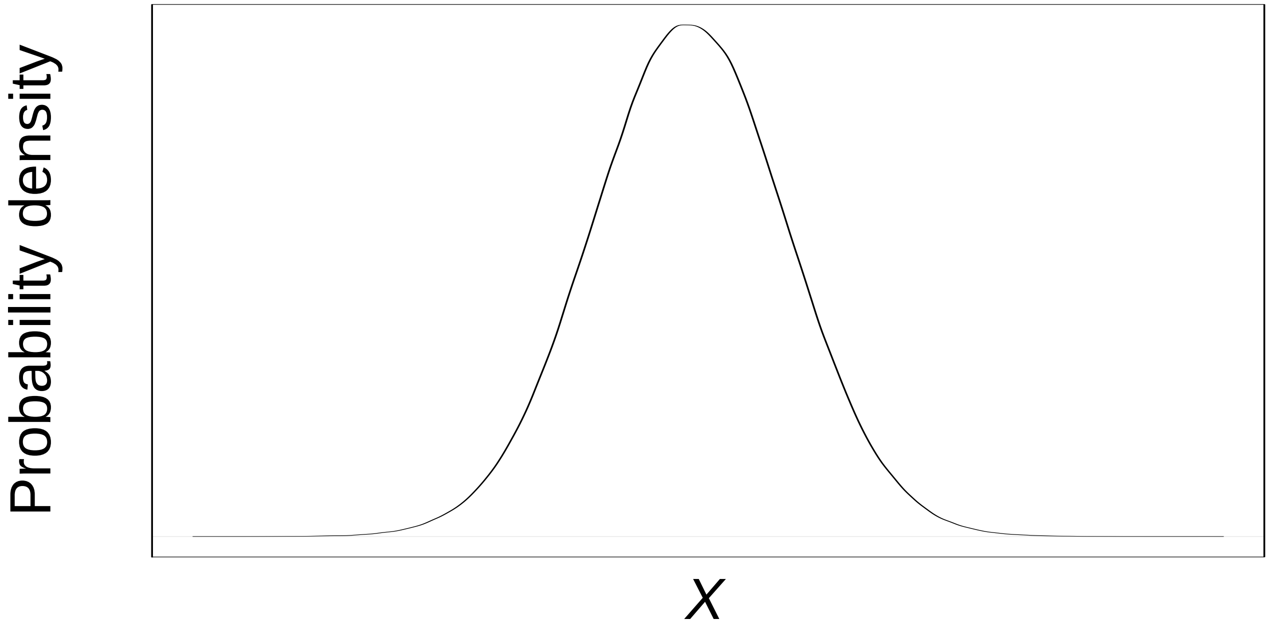
5.2.1 Introduction

Recall the “parameters” of the binomial and Poisson distributions:

- the **binomial distribution** has two parameters, n and p
- the **Poisson distribution** has one parameter λ
- The **Normal distribution** has two parameters: the mean, μ, and the standard deviation, σ

5.2.1 Introduction

It's probability density function (pdf) has a "**bell–shaped**" profile (page 125):



5.2.1 Introduction

The (rather nasty!) formula for this pdf is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$

Unlike the binomial and Poisson distributions, there is **no simple formula** for calculating probabilities.

Don't worry though, probabilities from the Normal distribution can be determined using **statistical tables** (see the end of this chapter) or statistical packages such as `Minitab`.

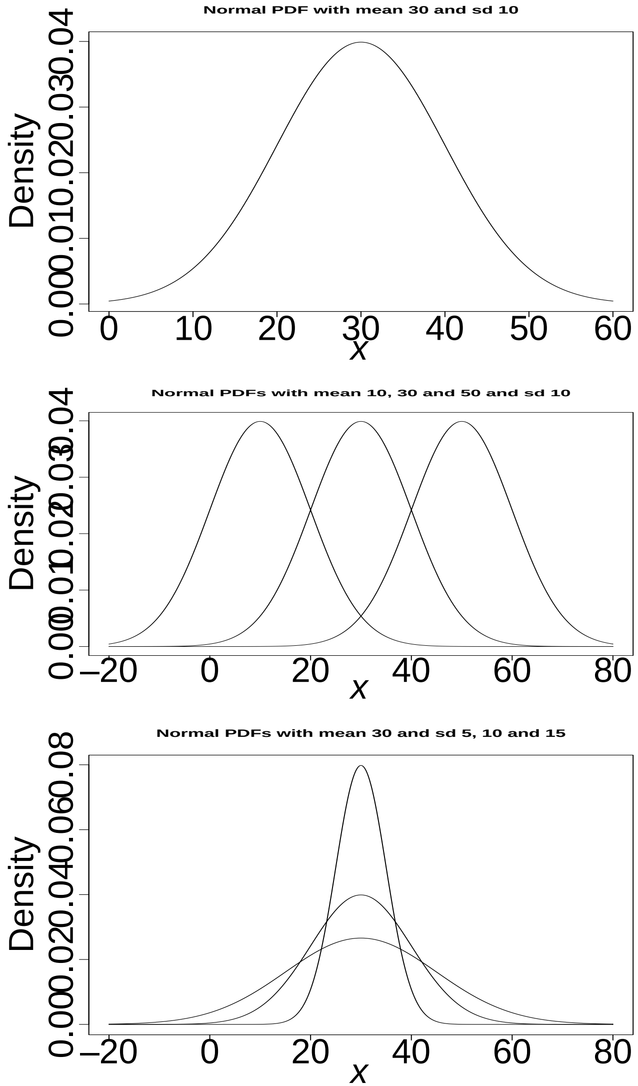
5.2.1 Introduction

There are four important characteristics of the Normal distribution:

1. It is **symmetrical** about its mean, μ.
2. The mean, median and mode all **coincide**.
3. The area under the curve is equal to 1.
4. The curve extends in both directions to infinity (∞).

On the next slide are plots of the pdf for Normal distributions with different values of μ and σ.

5.2.1 Introduction



5.2.2 Notation

If a random variable X has a Normal distribution with mean μ and variance σ^2, then we write

$$X \sim N\left(\mu, \sigma^2\right).$$

For example, a random variable X which follows a Normal distribution with mean **10** and variance **25** is written as

$$X \sim N(10, 25) \quad \text{or}$$
$$X \sim N\left(10, 5^2\right).$$

It is important to note that the second parameter in this notation is the **variance** and not the **standard deviation**.

5.2.3 The *standard* Normal distribution

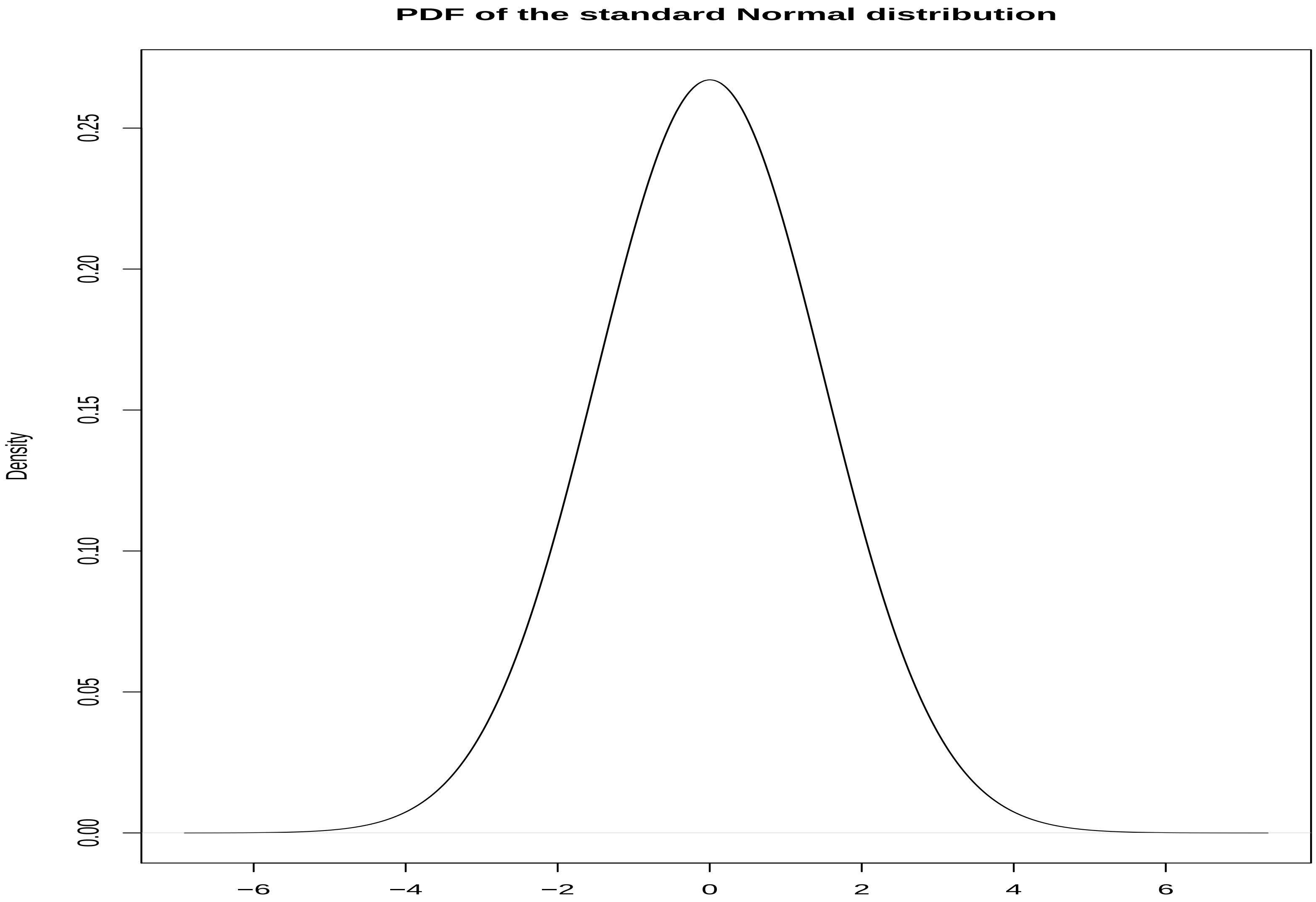
The **Standard** Normal distribution has a mean of **0** and a variance of **1**.

A random variable with this *standard* Normal distribution is usually given the letter Z, and so we say

$$Z \sim N(\mathbf{0}, \mathbf{1}).$$

If our random variable follows a **standard** Normal distribution, then we can obtain *cumulative probabilities* from statistical tables (see the table at the end of this chapter, which give "**less than or equal to**" probabilities).

5.2.3 Probability density function for Z



5.2.3 The *standard* Normal distribution

For example, if $Z \sim N(0, 1)$:

1. The probability that Z is less than -1.46 is $P(Z < -1.46)$. Therefore we look for the probability in tables corresponding to $z = -1.46$: row labelled -1.4, column headed -0.06.
   This gives $P(Z < -1.46) =$ **0.0721**.

2. The probability that Z is less than -0.01 is $P(Z < -0.01)$. Therefore we look for the probability in tables corresponding to $z = -0.01$: row labelled 0.0, column headed -0.01.
   This gives $P(Z < -0.01) =$ **0.4960**.

3. Similarly, $P(Z < 0.01) =$ **0.5040**.

5.2.3 The *standard* Normal distribution

z	-0.09	-0.08	-0.07	-0.06	-0.05	-0.04	-0.03	-0.02	-0.01	0.00
-2.9	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019
-2.8	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026
-2.7	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035
-2.6	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047
-2.5	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062
-2.4	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082
-2.3	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107
-2.2	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139
-2.1	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179
-2.0	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228
-1.9	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287
-1.8	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359
-1.7	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446
-1.6	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548
-1.5	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668
-1.4	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793	0.0808
-1.3	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951	0.0968
-1.2	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131	0.1151
-1.1	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335	0.1357
-1.0	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562	0.1587
-0.9	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814	0.1841
-0.8	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090	0.2119
-0.7	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389	0.2420
-0.6	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709	0.2743
-0.5	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050	0.3085
-0.4	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409	0.3446
-0.3	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783	0.3821
-0.2	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168	0.4207
-0.1	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562	0.4602
-0.0	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960	0.5000

5.2.3 The *standard* Normal distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

5.2.3 The *standard* Normal distribution

So far so good? Hopefully! But what if we want a "**greater than**" probability?

These tables only give "**less than**" probabilities!

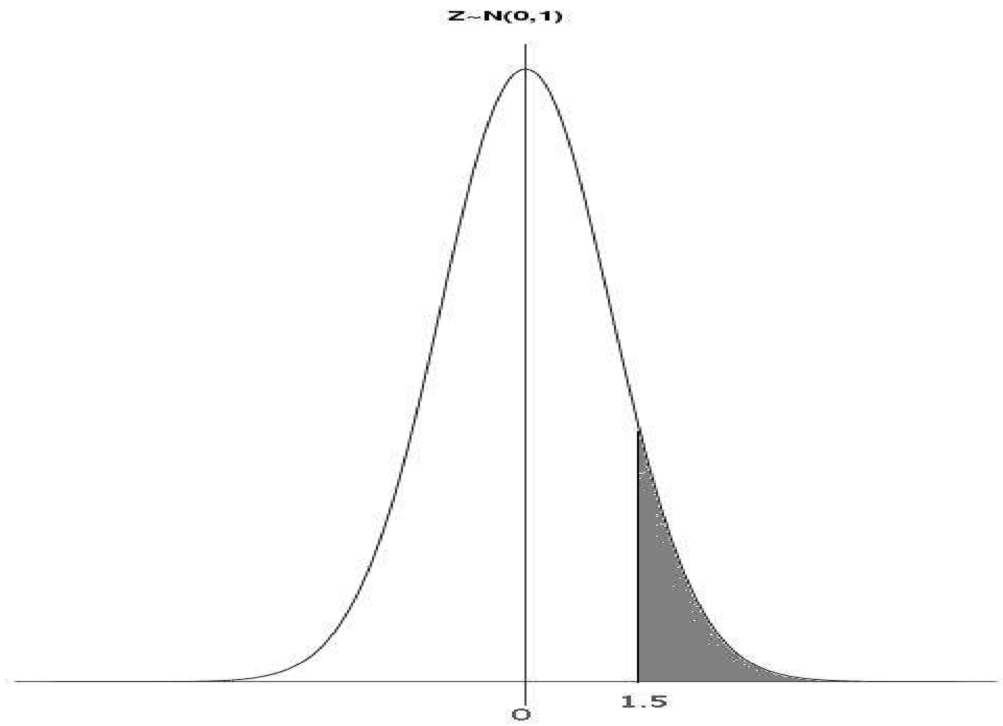
5.2.3 The *standard* Normal distribution

Easy! Remember,

- The area under the entire curve is equal to 1
- So we could find the "**less than**" probability and then subtract from 1 to get what's left over!

5.2.3 The *standard* Normal distribution

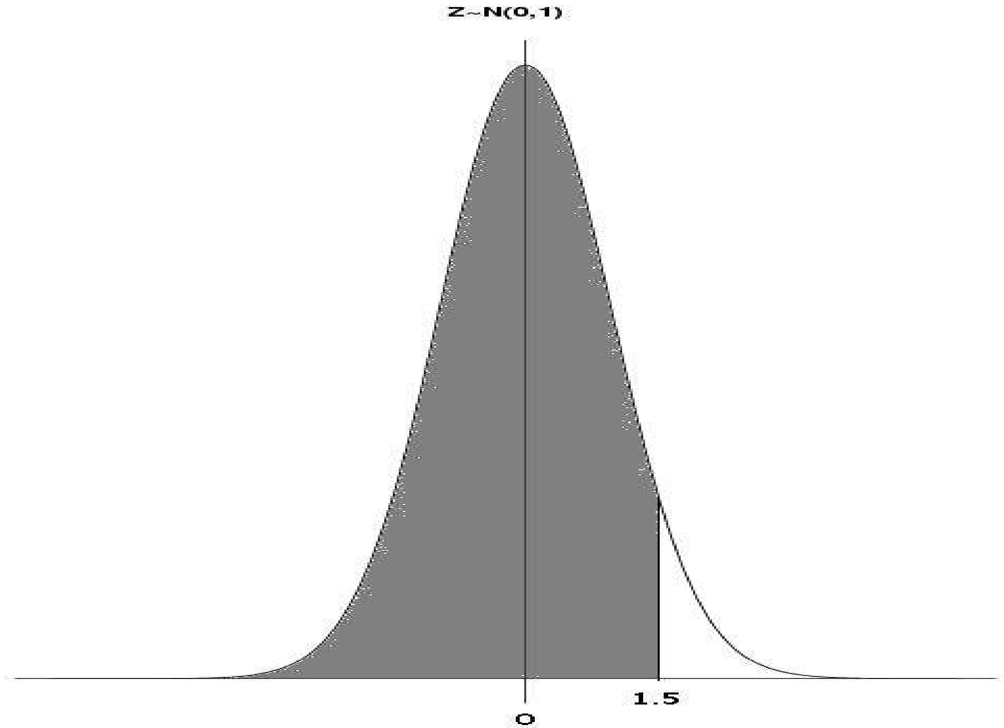
4. The probability that Z is greater than 1.5 is $P(Z > 1.5)$. Now our tables give "less than" probabilities, and here we want a "greater than" probability.



So we find $P(Z < 1.5) = 0.9332$ and subtract this from 1 to give **0.0668**.

5.2.3 The *standard* Normal distribution

5. What about the probability that Z lies between -1.2 and 1.5? It often helps to think about this graphically.



5.2.3 The *standard* Normal distribution

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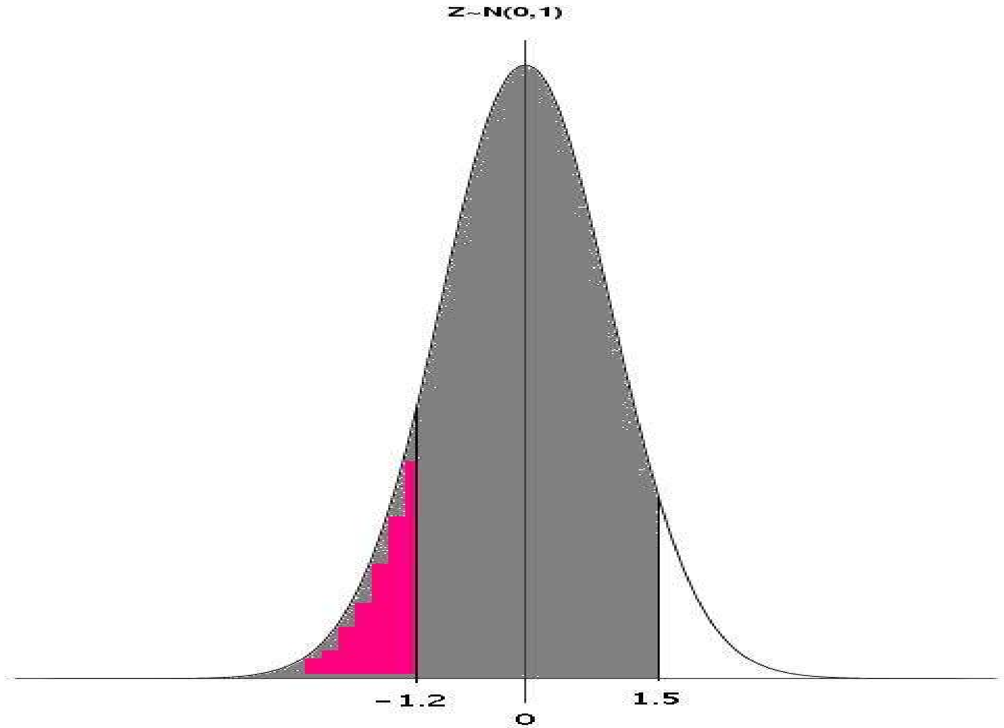
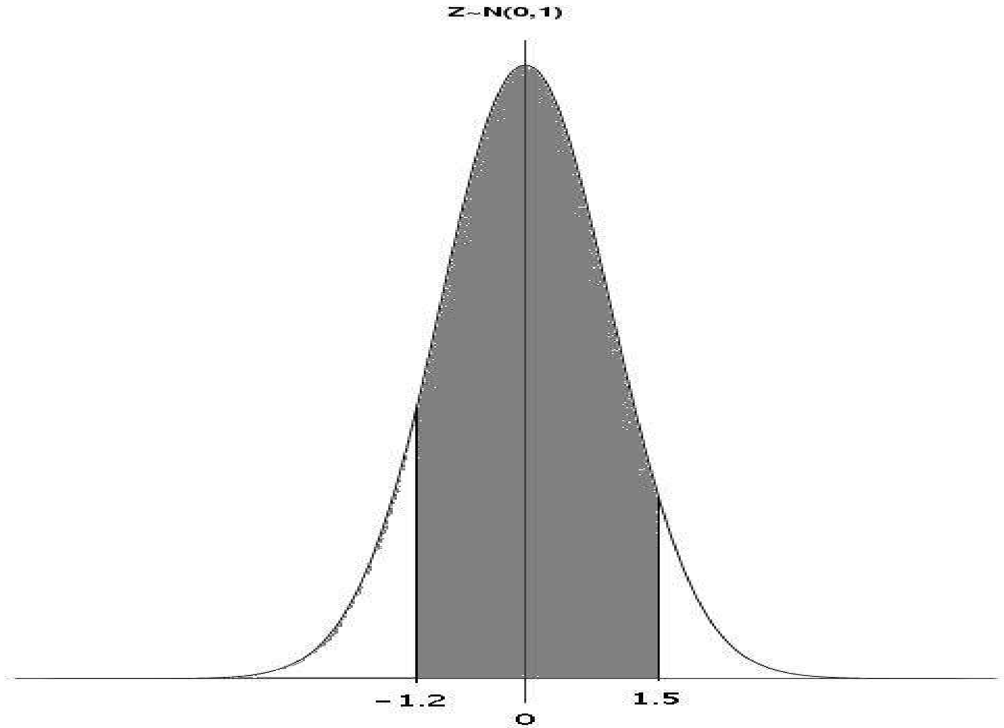


5. What about the probability that Z lies between -1.2 and 1.5? It often helps to think about this graphically.



Doing so, gives

$$P(-1.2 < Z < 1.5) = P(Z < 1.5) - P(Z \leq -1.2)$$
$$= 0.9332 - 0.1151$$

5.2.3 The *standard* Normal distribution

So how do we calculate probabilities for **any** Normal distribution, not just the **standard** Normal distribution – for which we have tables?

Idea: "make" the Normal distribution that we have "look like" the standard Normal distribution, and then we can just use the tables as before!

But how? Use the **slide–squash** technique!!

Example: IQ of graduates

The formula which changes **any** Normal random variable X into the **standard** Normal random variable Z is given by

$$Z = \frac{X - \mu}{\sigma},$$

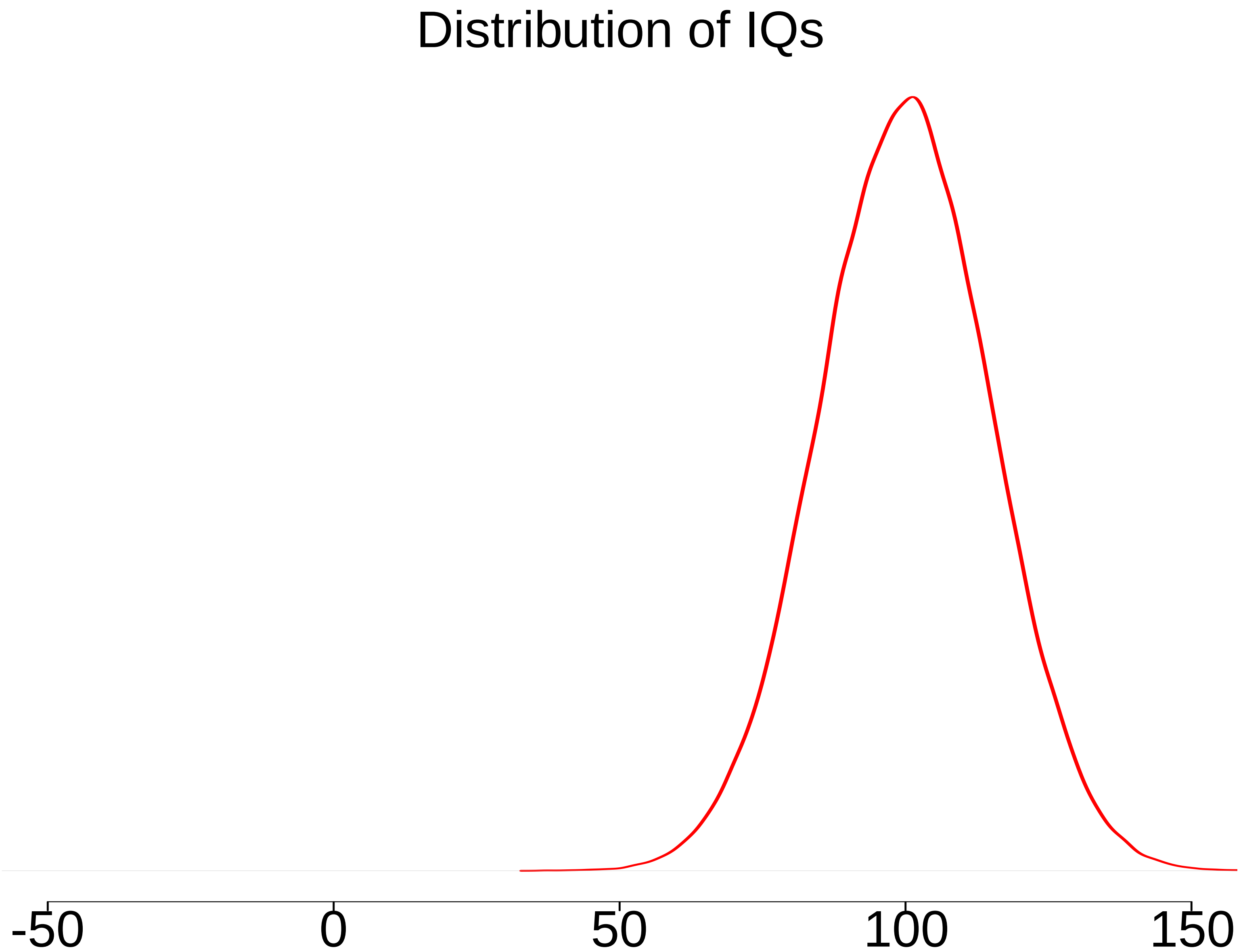
where

- μ is the mean
- σ is the standard deviation

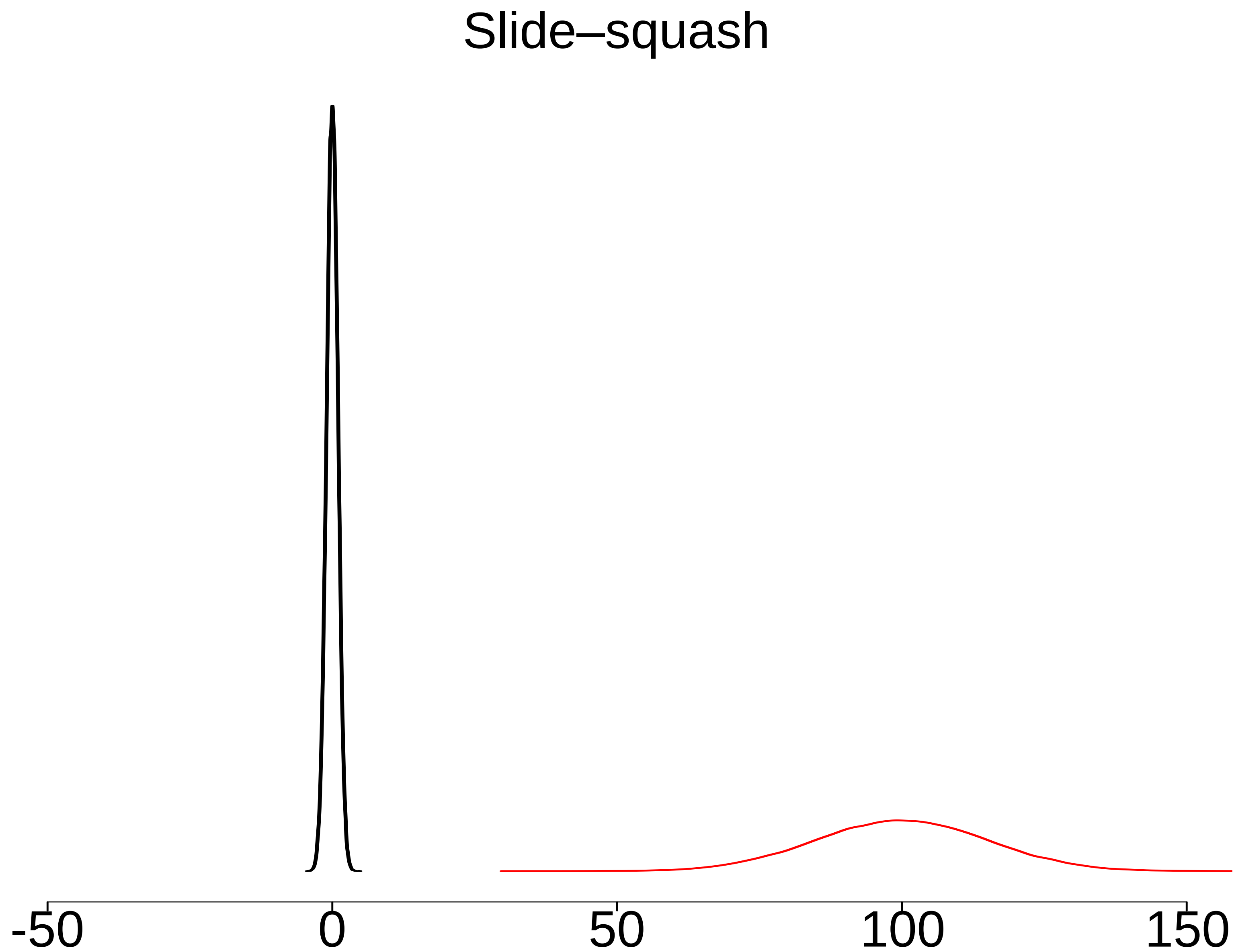
This can be translated into probability statements:

$$P(X \leq x) = P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

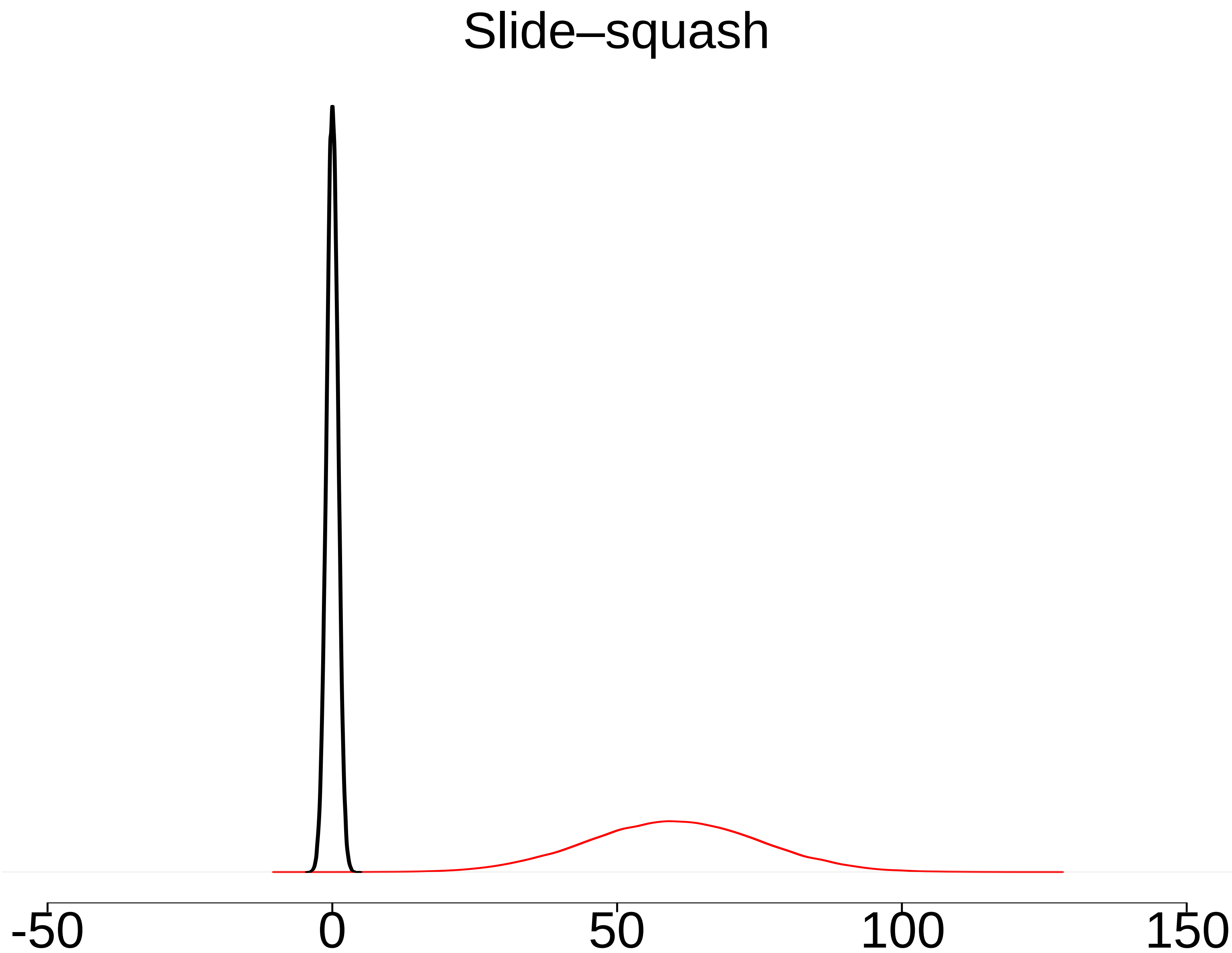
Example: IQ of graduates



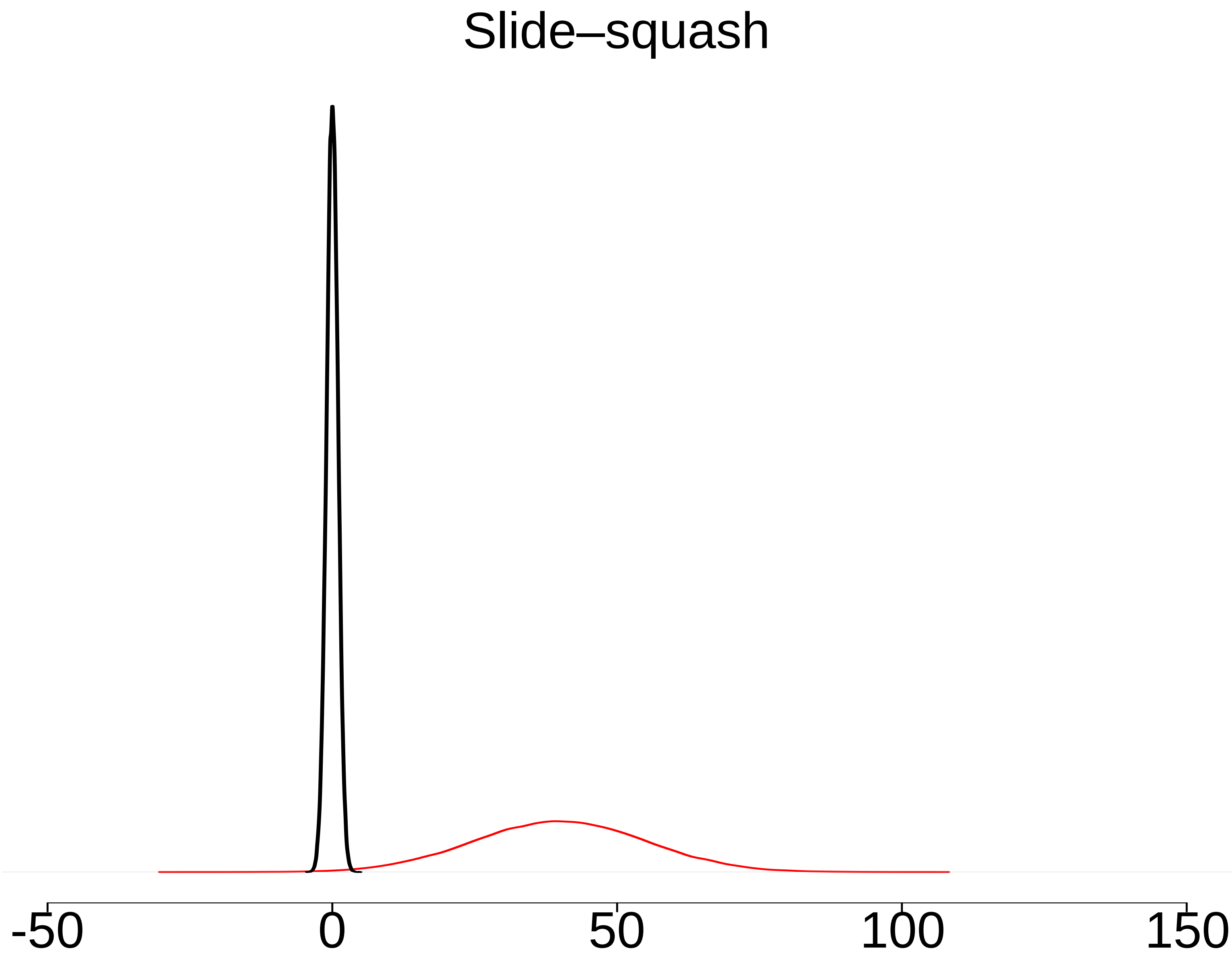
Example: IQ of graduates



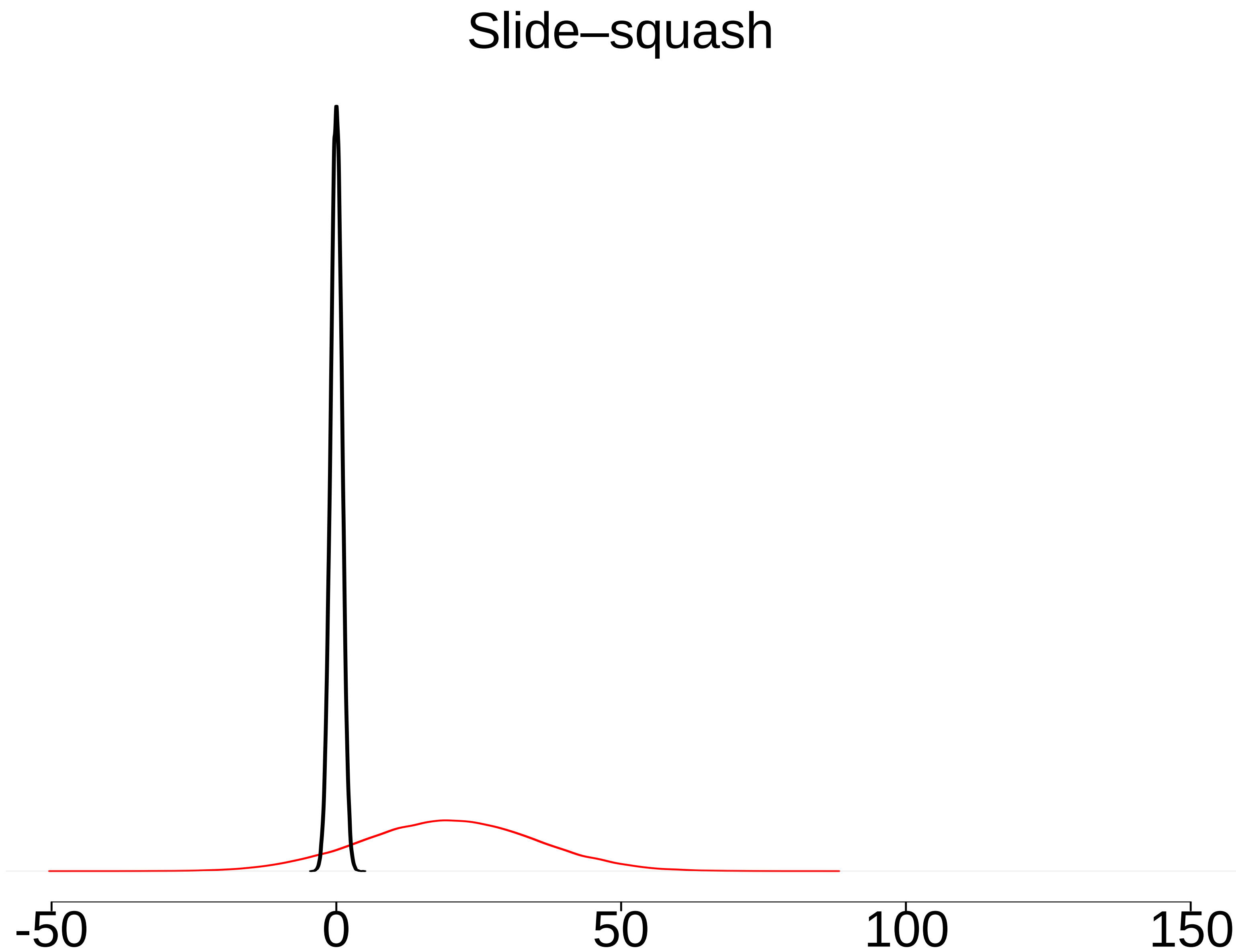
Example: IQ of graduates



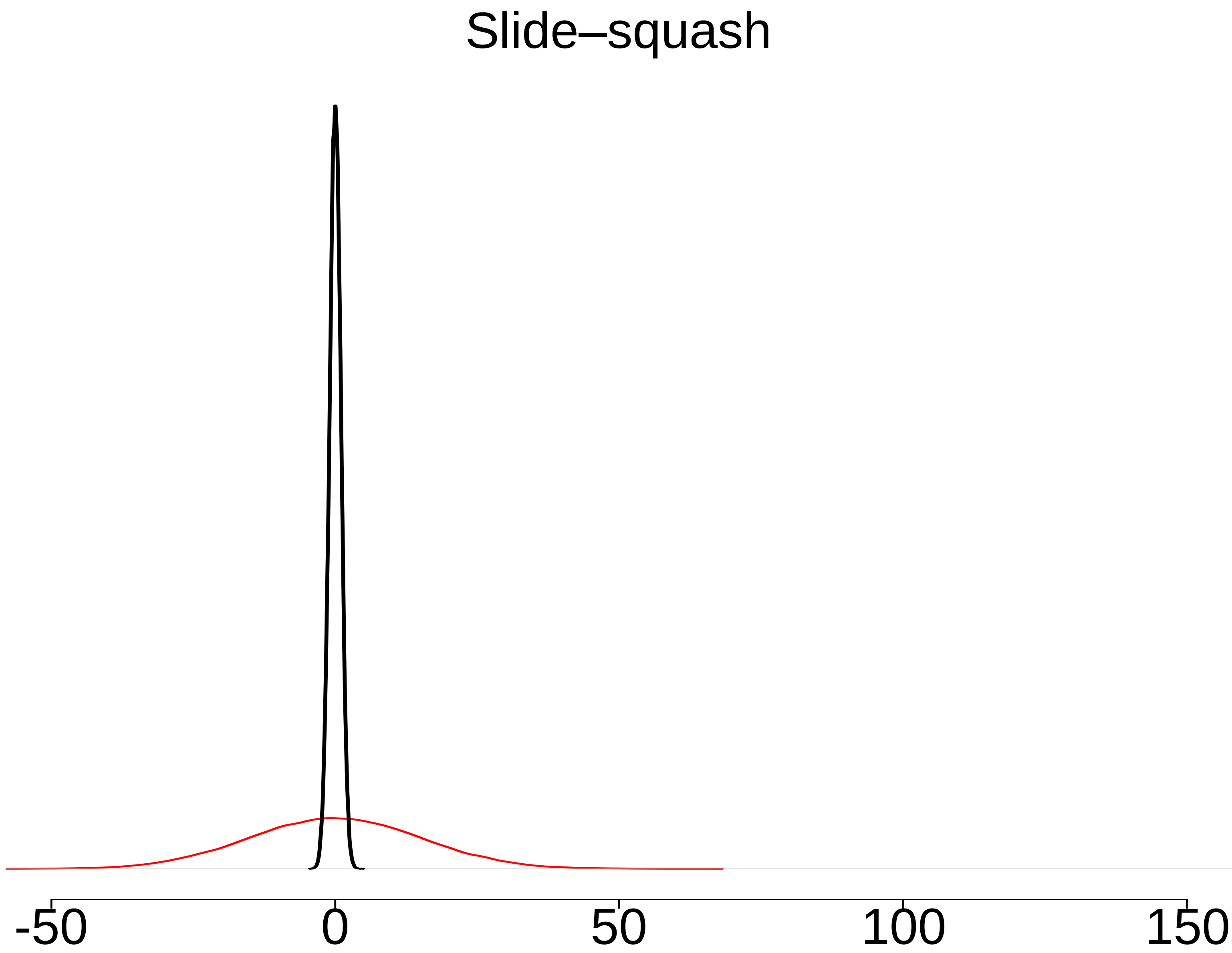
Example: IQ of graduates



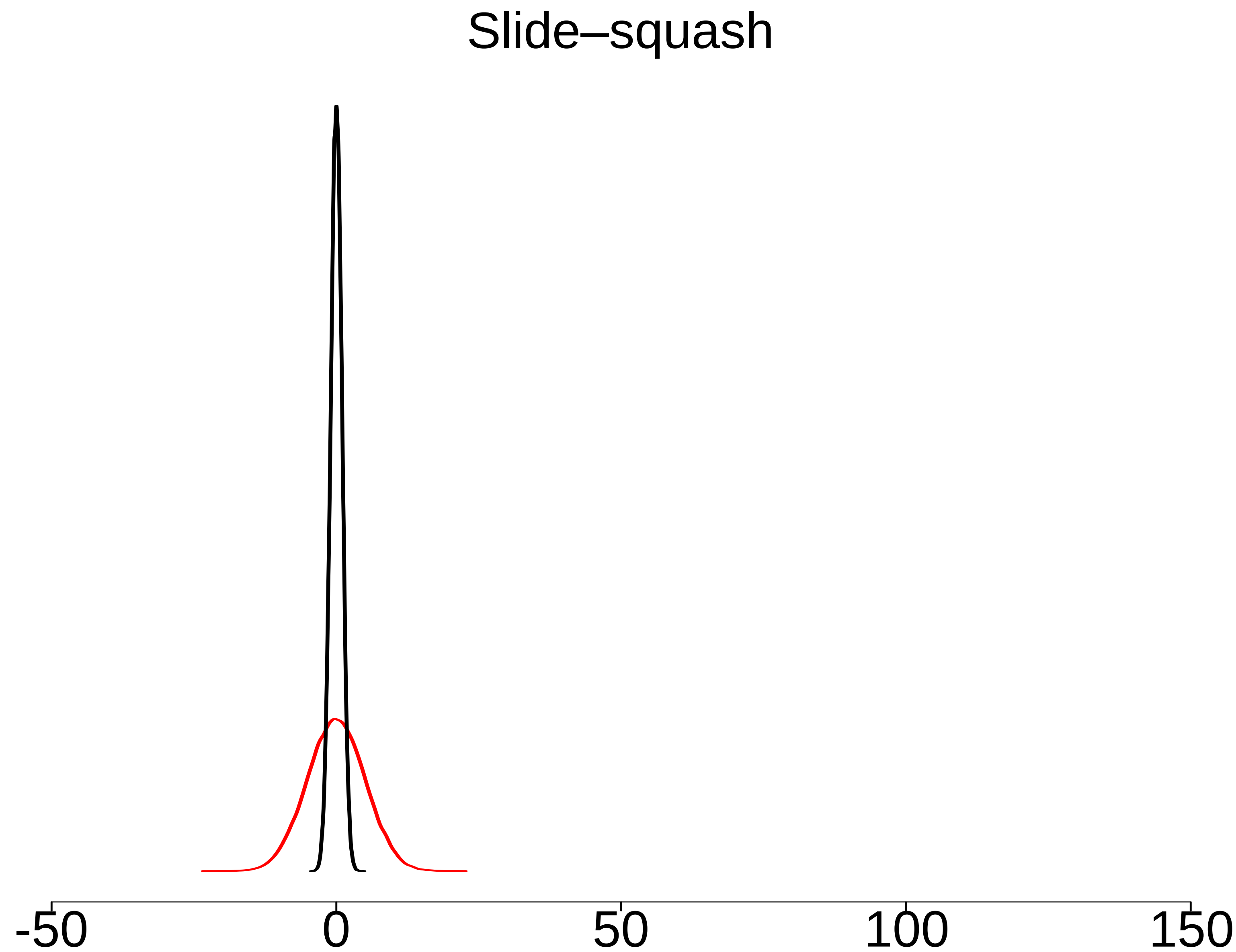
Example: IQ of graduates



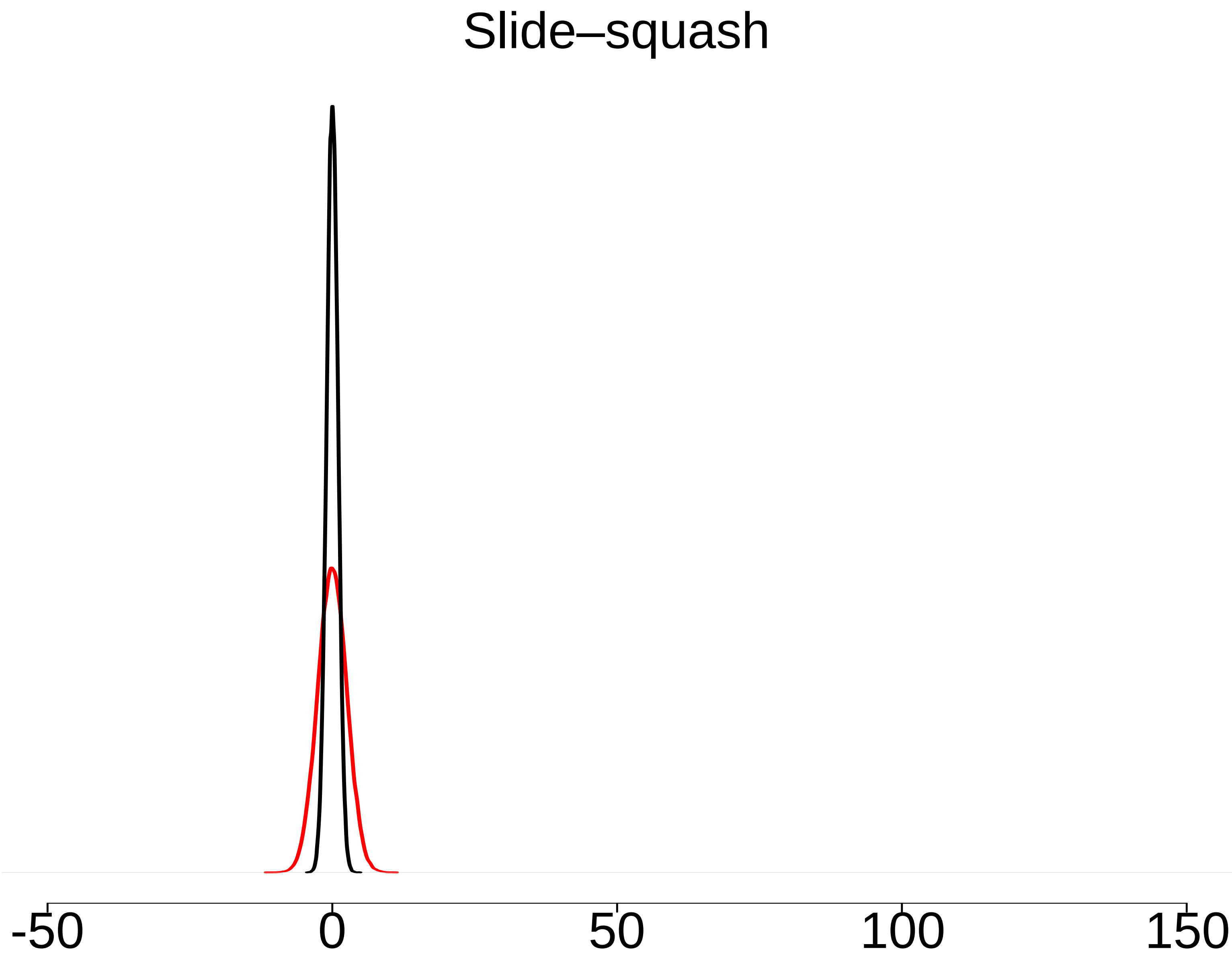
Example: IQ of graduates



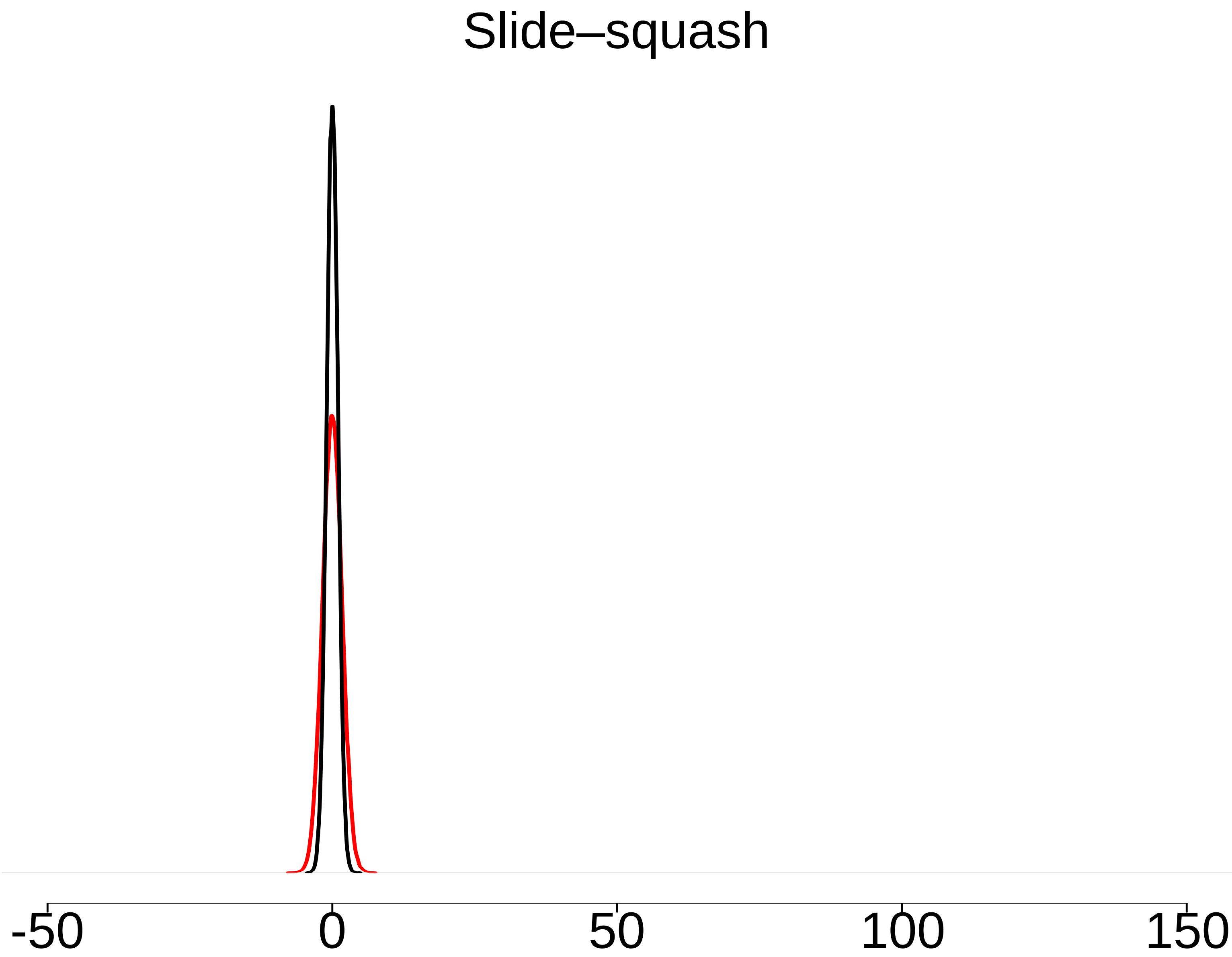
Example: IQ of graduates



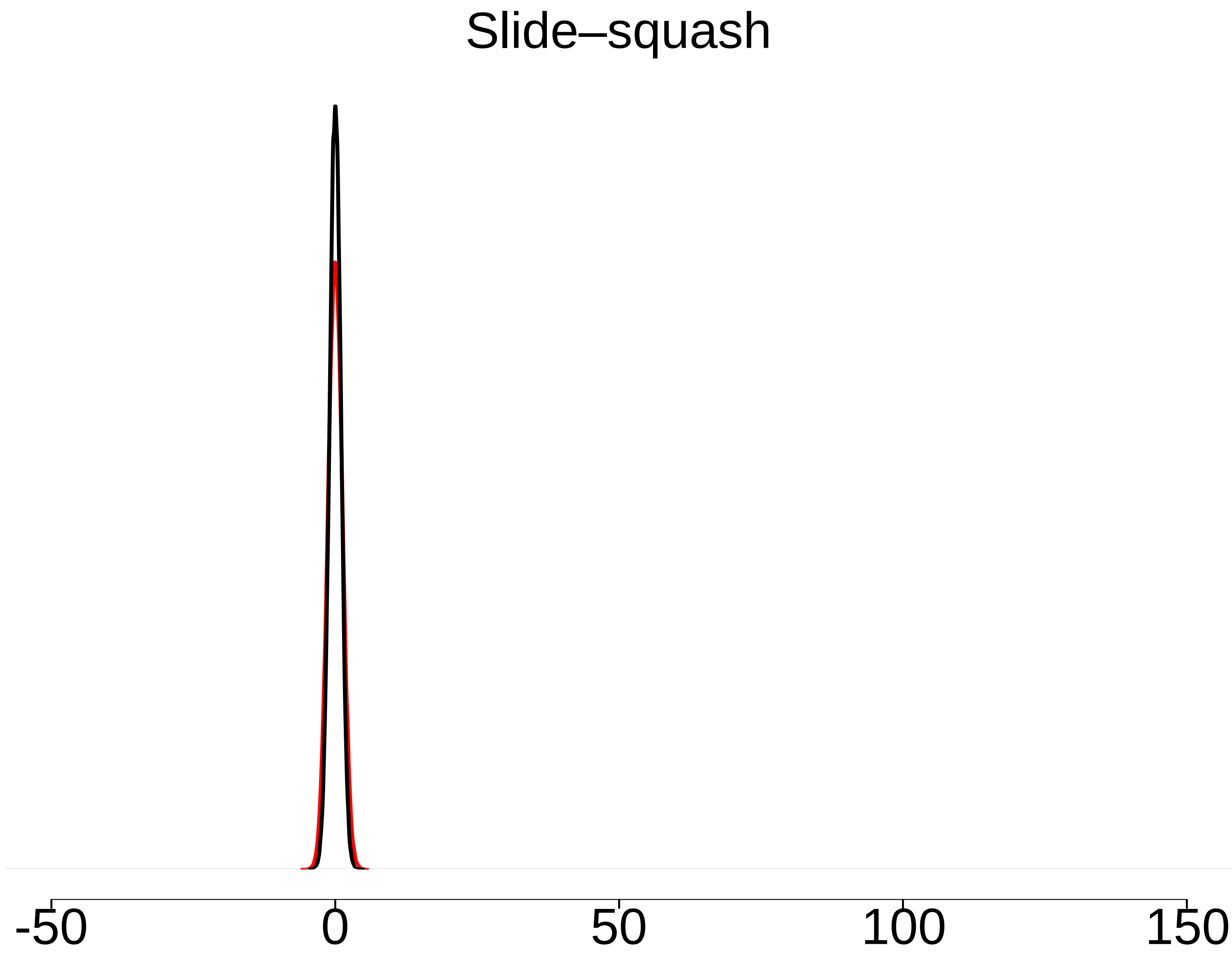
Example: IQ of graphs



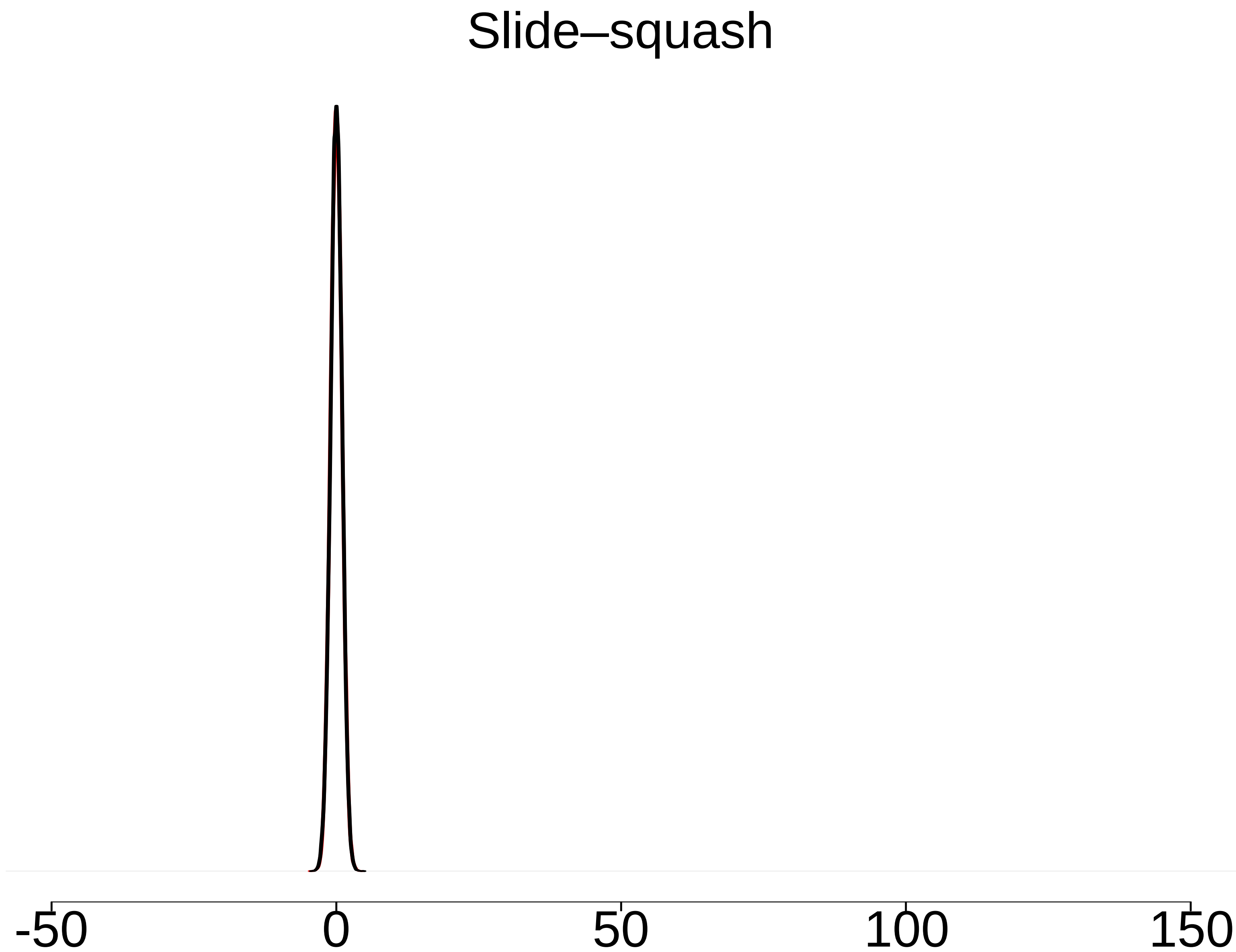
Example: IQ of graduates



Example: IQ of graduates



Example: IQ of graduates



Example: IQ of graduates

What is the probability that a graduate applicant has an IQ less than 85?

$$
\begin{aligned}
P(X < 85) &= P\left(Z < \frac{X - \mu}{\sigma}\right) \\
&= P\left(Z < \frac{85 - 100}{15}\right) \\
&= P(Z < -1) \\
&= 0.1587.
\end{aligned}
$$

Example: Solution to (i)

$$\begin{aligned} P(X < 110) &= P\left(Z < \frac{X - \mu}{\sigma}\right) \\ &= P\left(Z < \frac{110 - 100}{15}\right) \\ &= P(Z < 0.67) \\ &= 0.7486. \end{aligned}$$

Example: Solution to (ii)

$$
\begin{aligned}
P(X > 110) &= 1 - P(X < 110) \\
&= 1 - 0.7486 \\
&= 0.2514.
\end{aligned}
$$

Example: Solution to (iii)

$$
\begin{aligned}
P(X > 125) &= 1 - P(X < 125) \\
&= 1 - P\left(Z < \frac{125 - 100}{15}\right) \\
&= 1 - P(Z < 1.67) \\
&= 1 - 0.9525 \\
&= 0.0475.
\end{aligned}
$$

Example: Solution to (iv)

$$\begin{aligned}
P(95 < X < 115) &= P(X < 115) - P(X < 95) \\
&= P\left(Z < \frac{115-100}{15}\right) - P\left(Z < \frac{95-100}{15}\right) \\
&= P(Z < 1) - P(Z < -0.33) \\
&= 0.8413 - 0.3707 \\
&= 0.4706.
\end{aligned}$$

Example 5.1

You work as part of the design team for a social networking website. You are interested in the amount of time, X seconds, it takes users of your website to download a video.

A random sample of downloads gives a mean download time of 7 seconds, with a standard deviation of 2 seconds.

What is the probability that it will take a user of your website in the future

(a) more than 8.5 seconds;

(b) less than 2.5 seconds;

(c) between 2.5 and 8.5 seconds

to download this video?

Example 5.1(a): Solution

$$\begin{aligned} P(X > 8.5) &= P\left(Z > \frac{8.5 - 7}{2}\right) \\ &= P(Z > 0.75) \\ &= 1 - P(Z < 0.75) \\ &= 1 - 0.7734 = 0.2266. \end{aligned}$$

Example 5.1(b): Solution

$$
\begin{aligned}
P(X < 2.5) &= P\left(Z < \frac{2.5 - 7}{2}\right) \\
&= P(Z < -2.25) \\
&= 0.0122.
\end{aligned}
$$

Example 5.1(c): Solution

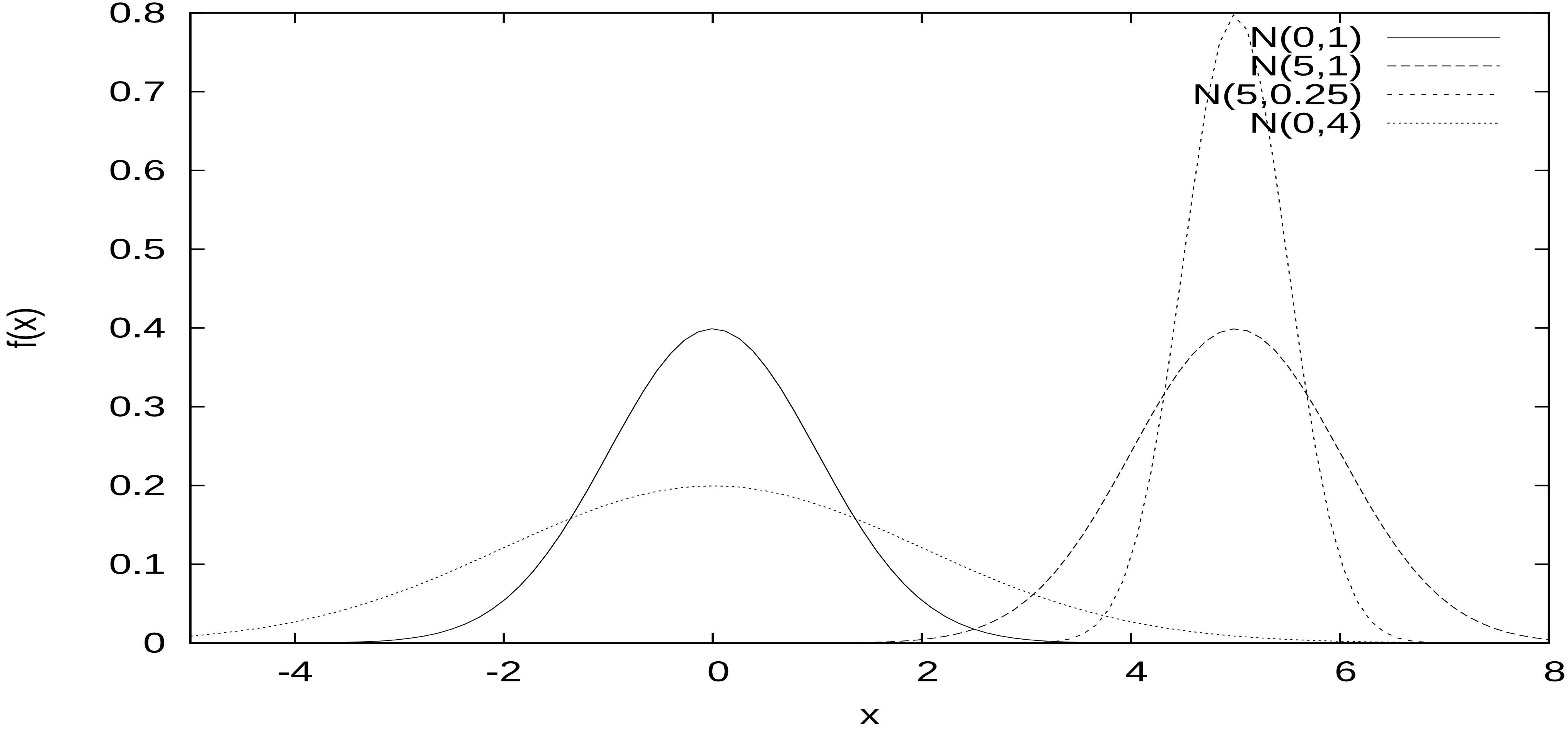
$$
\begin{aligned}
P(2.5 < X < 8.5) &= P(X < 8.5) - P(X < 2.5) \\
&= 0.7734 - 0.0122 = 0.2144.
\end{aligned}
$$

5.3 More Continuous Probability Models

Over the past few weeks we have discussed some “standard” probability distributions which can be used to model data.

We have looked at two such distributions for **discrete** data – the binomial distribution and the Poisson distribution – and last week the Normal distribution was introduced as a probability model for **continuous** data.

Recall the **probability density function** of the Normal distribution, which is often referred to as a "bell–shaped curve":



5.3 More Continuous Probability Models

We saw in the lecture last week that many naturally occurring continuous measurements (such as height, weight, time, rainfall etc.) often resemble this bell–shaped curve when plotted using a histogram, for example.

But what if we cannot assume “Normality” for our data?

We now consider two other probability models which can be used to model continuous data when the Normal distribution isn’t appropriate.

5.3 Example of “non–Normality”

- You manage a group of Environmental Health Officers and need to decide at what time they should inspect a local hotel
- You decide that any time during the working day (9.00 to 18.00) is okay
- You want to decide the time “randomly”

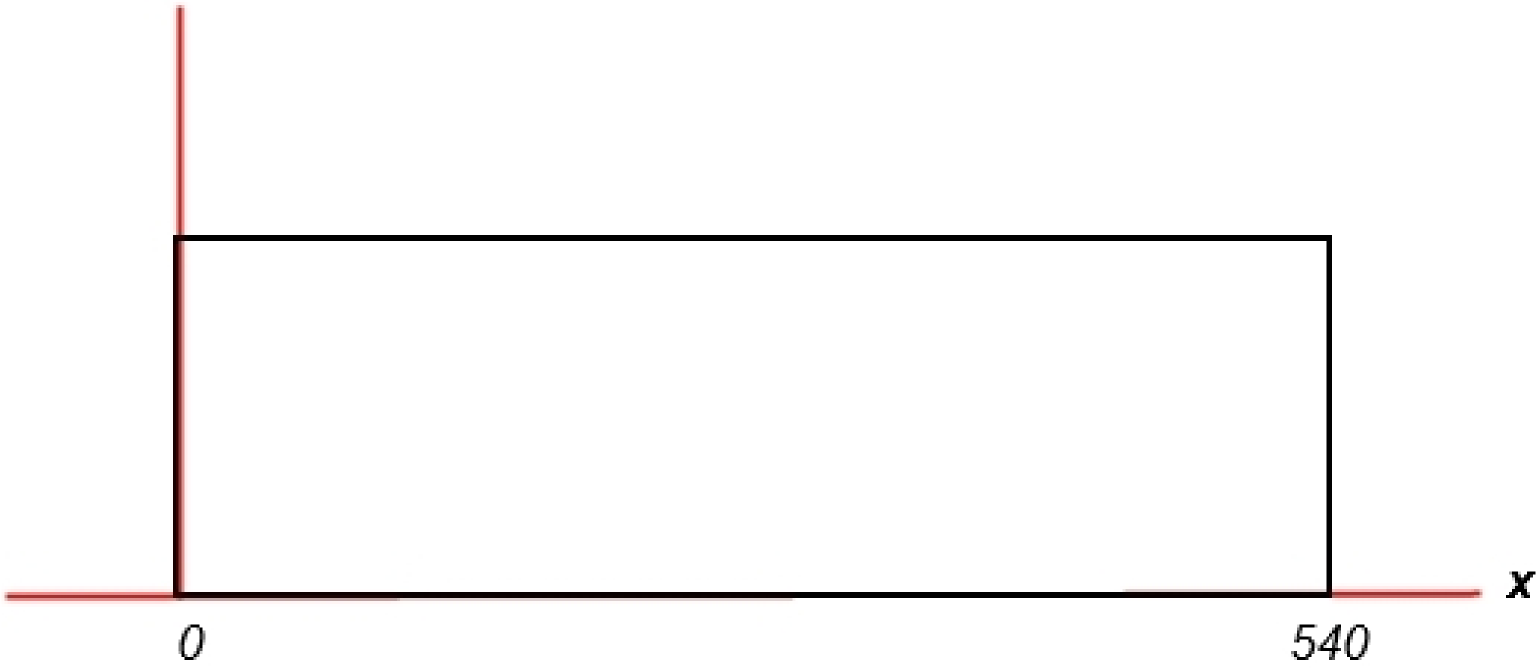
Here, “**randomly**” is a short–hand for

“*a random time, where all times in the working day are equally likely to be chosen*”

5.3.1 The Uniform distribution

Let X be the time to their arrival at the hotel, measured in terms of minutes from the start of the day.

Then X is a **Uniform** random variable between 0 and 540 (page 134):



5.3.1 The Uniform distribution

As with the Normal distribution, the total area (base $\times$ height) under the pdf must equal one.

Therefore, as the base is 540, the height must be $1/540$.

Hence the **probability density function** (pdf) for the continuous random variable X is

$$f(x) = \begin{cases} \dfrac{1}{540} & \text{for } 0 \leq x \leq 540 \\ 0 & \text{otherwise.} \end{cases}$$

5.3.1 The Uniform distribution

In general, we say that a random variable X which is equally likely to take any value between a and b has a **uniform distribution** on the interval $\boldsymbol{a}$ to $\boldsymbol{b}$, i.e.

$$X \sim U(a, b).$$

5.3.1 The Uniform distribution

The random variable has **probability density function** (pdf)

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and probabilities can be calculated using the formula

$$P(X \leq x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b. \end{cases}$$

5.3.1 The Uniform distribution

Therefore, for example, the **probability that the inspectors visit the hotel in the morning** (within 180 minutes after 9am) is

$$P(X \leq 180) = \frac{180 - 0}{540 - 0} = \frac{1}{3}.$$

5.3.1 The Uniform distribution

The **probability of a visit during the lunch hour** (12.30 to 13.30) is

$$
\begin{aligned}
P(210 \leq X \leq 270) &= P(X \leq 270) - P(X < 210) \\
&= \frac{270 - 0}{540 - 0} - \frac{210 - 0}{540 - 0} \\
&= \frac{270 - 210}{540} \\
&= \frac{60}{540} \\
&= \frac{1}{9}.
\end{aligned}
$$

5.3.1 Uniform distribution: Mean and Variance

Recall that:

- If $X \sim \text{bin}(n, p)$, then
  - $E(X) = n \times p$ and
  - $\text{Var}(X) = n \times p \times (1 - p)$
- If $X \sim \text{Po}(\lambda)$, then
  - $E(X) = \lambda$ and
  - $\text{Var}(X) = \lambda$

5.3.1 Uniform distribution: Mean and Variance

We have equivalent formulae for $X \sim U(a, b)$:

$$E(X) = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}.$$

5.3.1 Uniform distribution: Mean and Variance

In the above example, we have

$$E(X) = \frac{a+b}{2} = \frac{0+540}{2} = 270,$$

so that the mean arrival of the inspectors is 9am + 270 minutes = 13.30.

Also

$$\text{Var}(X) = \frac{(540-0)^2}{12} = 24300,$$

and therefore $SD(X) = \sqrt{Var(X)} = \sqrt{24300} = 155.9$ minutes.

5.3.2 The Exponential distribution

The **exponential distribution** is another common distribution that is used to describe continuous random variables.

It is often used to model lifetimes of products and times between “random” events, for example:

- Lifetime of light bulbs
- Arrival of customers in a queueing system
- Arrival of orders

The distribution has one parameter, λ. If our random variable X follows an **exponential distribution**, then we say

$$X \quad \sim \quad \exp(\lambda).$$

5.3.2 The Exponential distribution

Its **probability density function** is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

and probabilities can be calculated using

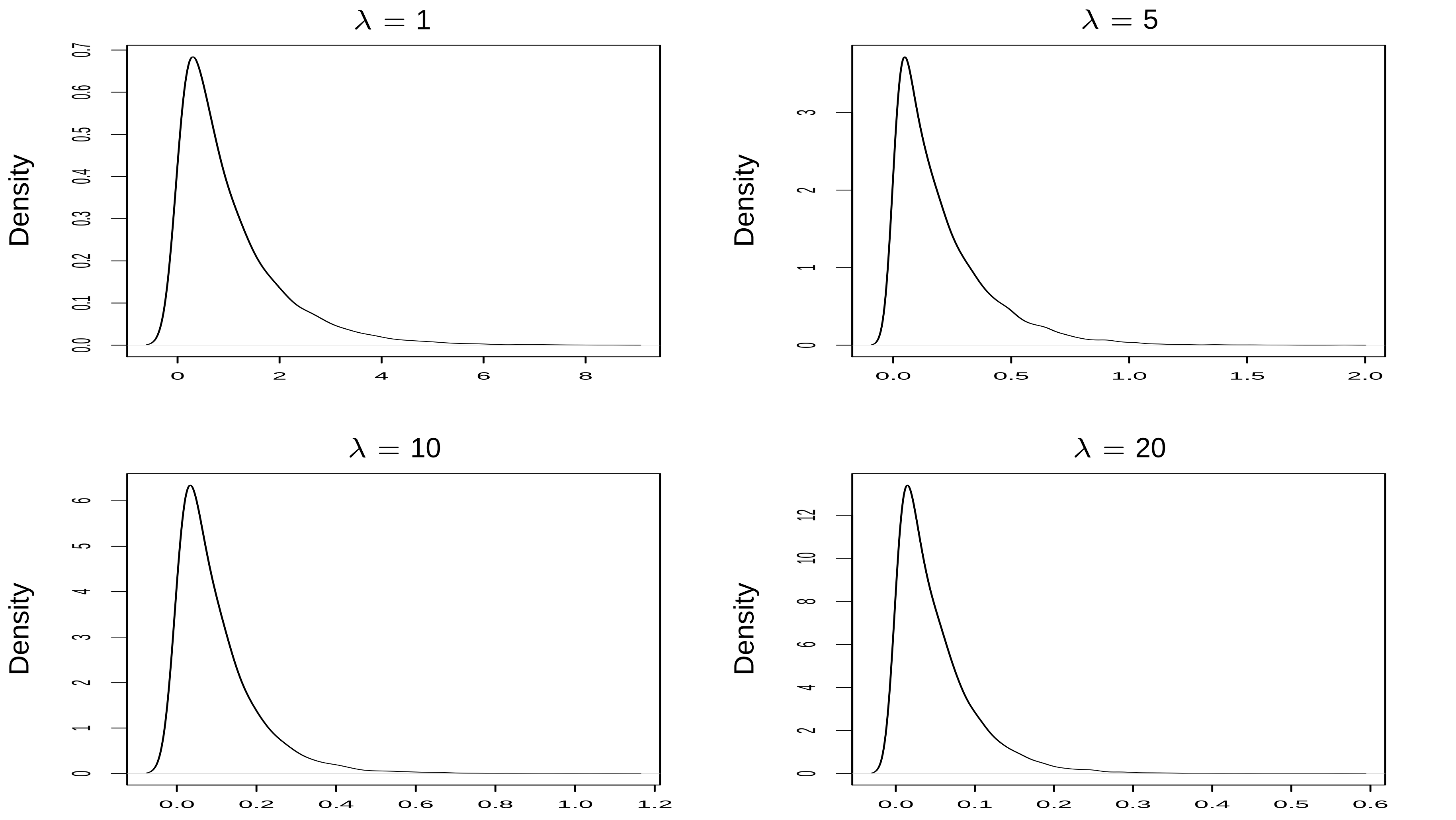
$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x > 0. \end{cases}$$

5.3.2 The Exponential distribution

The main **features** of this distribution are:

1. an exponentially distributed random variable can only take **positive** values
2. larger values are **increasingly unlikely** – "**exponential decay**"
3. the value of λ fixes the **rate of decay** – *larger values correspond to more rapid decay*.

5.3.2 The Exponential distribution



5.3.2 The Exponential distribution

Consider an example in which the time (in minutes) between successive users of a self–service supermarket checkout can be modelled by an exponential distribution with $\lambda = 0.3$.

The probability of the gap between users being less than 5 minutes is

$$
\begin{aligned}
P(X < 5) &= 1 - e^{-0.3\times 5} \\
&= 1 - 0.223 = 0.777.
\end{aligned}
$$

Also the probability that the gap is more than 10 minutes is

$$
\begin{aligned}
P(X > 10) &= 1 - P(X < 10) \\
&= 1 - \left(1 - e^{-0.3\times 10}\right) \\
&= e^{-0.3\times 10} = 0.050
\end{aligned}
$$

and the probability that the gap is between 5 and 10 minutes is

$$
\begin{aligned}
P(5 < X < 10) &= P(X < 10) - P(X < 5) \\
&= 0.950 - 0.777 = 0.173.
\end{aligned}
$$

5.3.2 The Exponential distribution

One of the main uses of the exponential distribution is as a model for the **times between events occurring randomly in time**.

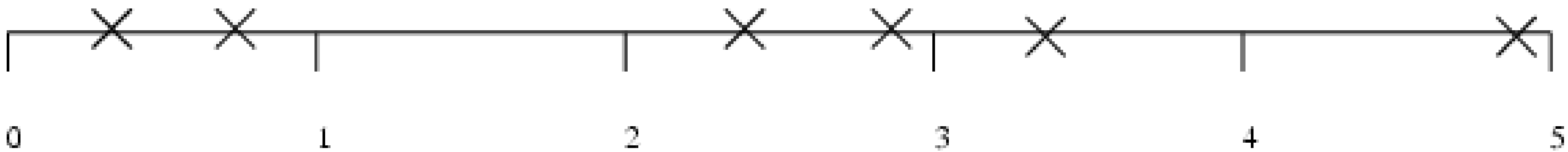
We have previously considered events which occur at random points in time in connection with the **Poisson distribution**.

The Poisson distribution describes probabilities for the number of events taking place in a given time period.

The exponential distribution describes probabilities for the times between events. Both of these concern events occurring randomly in time (at a constant average rate, say λ). This is known as a **Poisson process**.

5.3.2 The Exponential distribution

Consider a series of randomly occurring events such as calls at a credit card call centre. The times of calls might look like



We can view these data in **two ways**:

- The number of calls in each minute (here 2, 0, 2, 1 and 1)
- the times *between* successive calls

5.3.2 The Exponential distribution

For the **Poisson process**,

- the number of calls has a **Poisson** distribution with parameter λ, and
- the time between successive calls has an **exponential** distribution with parameter λ.

5.3.2 Exponential distribution: Mean and Variance

The mean and variance of the exponential distribution can be shown to be

$$E(X) = \frac{1}{\lambda}, \qquad \text{Var}(X) = \frac{1}{\lambda^2}.$$

Example 5.2

According to Apple's technical support site, `www.apple.com/support/itunes`, downloading an iPod game using a broadband connection should take 3 to 6 minutes.

Assuming that download times are uniformly distributed between 3 and 6 minutes, if you download a game what is the probability that the download time will be

(a) less than 3 minutes 15 seconds?

(b) More than 5 minutes?

(c) More than 7 minutes?

(d) What are the mean and standard deviation download times?

Example 5.2(a): Solution

Let X: Time taken to download a game (minutes). Then $X \sim U(3,6)$.

So

$$P(X < 3.25) \;=\; \frac{3.25 - 3}{6 - 3} = \frac{1}{12} = 0.0833.$$

Example 5.2(b),(c): Solution

Similarly,

$$\begin{aligned} P(X > 5) &= 1 - P(X < 5) \\ &= 1 - \frac{5-3}{6-3} = \frac{1}{3} = 0.3333. \end{aligned}$$

Also, $P(X > 7) = 0$... why?

Example 5.2(d): Solution

If $X \sim U(a, b)$,

$$E[X] = \frac{a+b}{2} \qquad \text{and} \qquad \text{Var}(X) = \frac{(b-a)^2}{12}.$$

So we have

$$E[X] = \frac{3+6}{2} = 4.5 \text{ minutes}$$

and

$$\text{SD}(X) = \sqrt{\frac{(6-3)^2}{12}} = 0.866 \text{ minutes}$$

Example 5.3

Customers arrive at the drive–through window of a fast food restaurant at a rate of 2 per minute during the lunch hour.

(a) What is the probability that the next customer will arrive within 1 minute?

(b) What is the probability that the next customer will arrive within 20 seconds?

(c) What is the mean time between arrivals at the drive–through window? What about the standard deviation?

Example 5.3(a),(b): Solution

Let X: Time between arrivals at the drive–through window (minutes); $X \sim \exp(2)$. Then

$$\begin{aligned} P(X < 1) &= 1 - e^{-2\times 1} \\ &= 0.865. \end{aligned}$$

Similarly,

$$\begin{aligned} P\left(X < \frac{1}{3}\right) &= 1 - e^{-2\times \frac{1}{3}} \\ &= 0.487 \end{aligned}$$

Example 5.3(c): Solution

If $X \sim \exp(\lambda)$, then

$$E[X] = \frac{1}{\lambda} \qquad \text{and} \qquad \text{Var}(X) = \frac{1}{\lambda^2}.$$

Thus, in our example,

$$E[X] = \frac{1}{2} = 0.5 \text{ minutes} = 30 \text{ seconds},$$

and

$$\text{SD}(X) = \sqrt{\frac{1}{2^2}} = \sqrt{\frac{1}{4}} = 30 \text{ seconds}.$$