



Summer Project

SCHOOL OF MATHEMATICS, STATISTICS & PHYSICS

Summer 2017

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Contents

1	Some revision	4
1.1	Quadratic equations	4
1.2	Plane geometry	4
1.3	Logarithms and exponentials	5
1.4	Trigonometric functions	5
1.5	Differentiation and integration	5
1.5.1	Differentiation rules	6
1.5.2	Integration rules	6
2	New topic 1: Modular arithmetic	7
2.1	Introduction	7
2.2	The modulo operator	8
2.2.1	Addition	9
2.2.2	Multiplication	10
3	New topic 2: Complex numbers	10
3.1	Introduction	10
3.2	The general complex number	12
3.3	Complex arithmetic	12
3.3.1	Addition and subtraction	12
3.3.2	Multiplication	13
3.4	The conjugate of a complex number	13
3.4.1	Division	14
4	Written assignment	15
5	Research question	17

Introduction

This project is intended to give current A-Level students (or equivalent) an opportunity to demonstrate their mathematical knowledge and experience, and to give such students a taste of some new material which most will not have seen before. A brief recap of some concepts encountered in the A-level syllabus is provided, with a more detailed description of new concepts to provide assistance when preparing solutions to questions posed in the project. This project consists of three components:

- A written assignment sheet
- Some computer based questions using our NUMBAS online testing system
- A more open ended, research-style question

The questions are designed to be of varying difficulty to challenge students at different skill levels and we would expect a good student to take up to five days to complete all components. Any questions marked with a “ * ” will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the information in this booklet carefully before attempting the questions. Completed assignments will be graded and successful students will be awarded with a certificate and general feedback. More details about this summer project can be found on our dedicated webpage: <http://www.mas.ncl.ac.uk/~nlf8/summer-project/>.

Written assignment

There are eight questions to work through here. Most of these questions will be based on material covered in this booklet. However, there may be one or two surprises! Solutions to these questions can be hand-written or typed. Hand-written solutions should be scanned and emailed to mathsandstats@ncl.ac.uk. Solutions prepared in a word-processing package (eg. Microsoft Word) can be emailed directly, although we recommend exporting to portable document format (PDF) first, to help prevent loss of formatting. Students must submit their solutions by **4pm, Friday 1st September**.

Computer based test

This uses Newcastle University’s *NUMBAS online testing system*. The test will be available in two modes; *PRACTICE* and *ASSESSED*. Students can access the practice test as many times as they like. Each time they open a test, the format of the questions within that particular test will remain the same, but the numbers might change. The assessed mode will be active for two weeks only, from **Monday 21st August–Friday 1st September**. Students must login and complete the test by **4pm, Friday 1st September**. Only **one** attempt will be allowed, but students can pause the test as often as they wish— there are no time constraints. The practice mode will be available throughout August and September. Students are strongly advised to work through our *Introduction to NUMBAS* first – available at <https://numbas.mathcentre.ac.uk/test-yourself/getting-started/>. A link to the actual test, and further instructions about submission of the test will be available on our webpage soon.

Research-style question

This would be the final part of the project. Students will be given a choice of 3–4 questions, from which they are required to select only **one**. We expect solutions to any of these questions to be one or two pages long. Students can pick from a range of topics for this question, including options from statistics or mechanics. These questions do not have strict guidelines and we encourage students to be creative with their answers. As with the written assignment, solutions to the open-ended questions can be either hand-written or electronically produced with a word processing package. Answers should be emailed to mathsandstats@ncl.ac.uk no later than **4pm, Friday 1st September**.

1 Some revision

1.1 Quadratic equations

A root of the function $f(x)$ is any value of x which solves $f(x) = 0$. For a quadratic polynomial where $f(x) = ax^2 + bx + c$, there is a maximum of two roots. These can be found via three different methods:

1. Factorisation:

The quadratic equation $ax^2 + bx + c = 0$ can sometimes be expressed in the form $(dx + e)(jx + k) = 0$.

This implies that the roots are $x = -\frac{e}{d}$ and $x = -\frac{k}{j}$.

2. Quadratic formula:

Substitute the respective values of a, b and c into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Completing the square (only when $a = 1$):

$$x^2 + bx + c = (x + b/2)^2 - (b/2)^2 + c = 0 \Rightarrow (x + b/2)^2 = (b/2)^2 - c.$$

1.2 Plane geometry

The equation of a straight line in *gradient form* is

$$y = mx + c,$$

where m is the *gradient* and c is the *y-intercept*. The equation of the straight line between points (x_1, y_1) and (x_2, y_2) in the xy -plane can be found using the formulae:

$$y - y_1 = m(x - x_1), \quad \text{where} \quad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

1.3 Logarithms and exponentials

Index form: $y = a^x$ *Logarithmic form:* $\log_a(y) = x$

Here is a list of useful logarithmic identities:

- $\log_a(x) + \log_a(y) = \log_a(xy)$
- $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$
- $\log_a(x^k) = k\log_a(x)$

The exponential function $f(x) = e^x$ is a special function because $\frac{df}{dx} = f(x)$. The natural logarithm is $\log_e$ and is often denoted $\ln$. If at any point in the exercises you encounter a log without a subscript then it can be assumed it means $\log_e$, not $\log_{10}$ as is sometimes the convention.

1.4 Trigonometric functions

You should all be familiar with $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ and how they work. We will exclusively use the angle (θ) given in radians to allow us to use the differentiation rules. Some important trigonometric identities are:

- $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\operatorname{cosec}(\theta) = \frac{1}{\sin(\theta)}$
- $\sec(\theta) = \frac{1}{\cos(\theta)}$
- $\cot(\theta) = \frac{1}{\tan(\theta)}$

1.5 Differentiation and integration

For a polynomial function $f(x) = ax^n$, the *first-order derivative* $\frac{df}{dx} = f'(x) = anx^{n-1}$. The inverse of this operation is *integration*, where for $f(x) = ax^n$ then

$$\int f(x) \, dx = \frac{a}{n+1}x^{n+1} + c,$$

where c is the constant of integration.

Here is a table of useful known derivatives and integrals,

Integral	Function	Derivative
$-\frac{1}{k}\cos(k\theta)$	$\sin(k\theta)$	$k\cos(k\theta)$
$\frac{1}{k}\sin(k\theta)$	$\cos(k\theta)$	$-k\sin(k\theta)$
$\frac{1}{k}e^{kx}$	e^{kx}	ke^{kx}
$x\ln(x) - x$	$\ln(x)$	$\frac{1}{x}$

1.5.1 Differentiation rules

The chain rule: The chain rule is used when differentiating a function of a function. So for a function $y(x) = f(g(x))$, we have:

$$\frac{dy}{dx} = \frac{dy}{d(g(x))} \times \frac{d(g(x))}{dx}.$$

The product rule: The product rule is used when differentiating a product of two functions. For a function of the form $y(x) = f(x)g(x)$, then:

$$\frac{dy}{dx} = \frac{df}{dx}g + f\frac{dg}{dx}, \quad \text{alternatively written as} \quad \frac{dy}{dx} = f'g + g'f,$$

where the $'$ denotes the derivative with respect to x .

The quotient rule: The quotient rule is used when differentiating a function divided by a function. For a function in the form $y(x) = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{u'v - v'u}{v^2},$$

again where the $'$ denotes a derivative with respect to x .

1.5.2 Integration rules

Integration by substitution: When integrating a function of the form

$$\int f(g(x))g'(x) dx,$$

we substitute $g(x) = u$ and $du = g'(x)dx$, which gives us

$$\int f(u)du.$$

Integration is then carried out normally with respect to u , before reverting back to x , as in the original function.

Integration by parts: Integration by parts is used when integrating a product of two functions. For $f(x) = u(x)v(x)$, the formula is:

$$\int uv \, dx = u \int v \, dx - \int u' \left(\int v \, dx \right) dx.$$

2 New topic 1: Modular arithmetic

We do not expect most students to have seen this topic before. Modular arithmetic is an extremely broad subject, filled with amazing proofs and mathematical techniques. We aim to simply introduce the basic computations using the modulo operator (as seen later). The purpose of including this new section is to give students a chance to familiarize themselves with certain mathematical computations which will be required of them should they pursue Mathematics beyond A-level. The content is kept as brief as possible, and many examples are included to encourage students to see for themselves how the general idea behind modular arithmetic works.

2.1 Introduction

Recall, When we divide a by b , where both a and b are integers and $b \neq 0$, we get the following:

$$\frac{a}{b} = q \text{ remainder } r,$$

where :

- a is the *dividend*,
- b is the *divisor*,
- q is the *quotient*,
- r is the *remainder*, where $0 \leq r < b$.

Under certain circumstances, we are interested only in the *remainder*.

2.2 The modulo operator

We will now introduce the modulo operator (*mod*).

For any integers a and b , we write $a(\bmod b)$ or $[a]_b$ to denote the remainder when a is divided by b .

Example 1: Find $17 \bmod 6$.

$$\frac{17}{6} = 2 \text{ remainder } 5$$

Remember, we are only interested in the *remainder* when using the modulo operator (*mod*). Thus, we have:

$$17 \bmod 6 \equiv 5.$$

Example 2: Find $-12 \bmod 17$.

We know that :

$$-12 = (-1) \cdot (17) + 5.$$

Thus,

$$-12 \bmod 17 \equiv 5.$$

2.2.1 Addition

When considering a sum within the modulo operator, we can first compute the modulo of each of the components individually and then take the modulo of the whole sum.

Example 3: Compute $[6 + 12 + 8]_5$.

It is clear that the expression within the modulo operator is a sum of three terms. We can take the modulo of each term individually, and finally take the modulo of the sum of the modulus.

We have: $[6]_5 = 1$, $[12]_5 = 2$, and $[8]_5 = 3$. Thus:

$$\begin{aligned}[6 + 12 + 8]_5 &= [[6]_5 + [12]_5 + [8]_5]_5 \\ &= [1 + 2 + 3]_5 \\ &= [6]_5 \\ &= 1.\end{aligned}$$

Let us verify our solution. We know the sum $6 + 12 + 8 = 26$. We can verify our solution by taking the remainder of 26 divided by 5.

$$\frac{26}{5} = 5 \text{ remainder } 1.$$

Therefore,

$$26 \pmod{5} \equiv 1.$$

2.2.2 Multiplication

Multiplication within the modulo operator works similarly.

Example 4: Compute $[2 \times 4 \times 8]_3$.

It is clear that the expression within the modulo operator is a product of three terms. We can take the modulo of each term individually, and finally take the modulo of the product of the modulus.

We have: $[2]_3 = 2$, $[4]_3 = 1$, and $[8]_3 = 2$.

Thus:

$$\begin{aligned}[2 \times 4 \times 8]_3 &= ([2]_3 \times [4]_3 + [8]_3)_3 \\ &= [2 \times 1 \times 2]_3 \\ &= [4]_3 \\ &= 1.\end{aligned}$$

Let us verify our solution. We know the product $2 \times 4 \times 8 = 64$. We can verify our solution by taking the remainder of 64 divided by 3.

$$\frac{64}{3} = 21 \text{ remainder } 1.$$

Therefore,

$$64 \pmod{3} \equiv 1.$$

3 New topic 2: Complex numbers

3.1 Introduction

To understand the next few sections, it would be helpful to be familiar with quadratic equations (eg. solving $x^2 - 3x + 2 = 0$; see section 1.1). In order to solve the equation of the form $ax^2 + bx + c = 0$, we can use the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

However, we know for certain that if x is a real number, i.e $x^2 \geq 0$, then there are some quadratic equations which do not have any real solutions.

Example 1: Solve the quadratic equation $x^2 + 6x + 13 = 0$.
Using the formula stated above with $a = 1$, $b = 6$ and $c = 13$,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{-16}}{2}. \end{aligned}$$

We can see that this equation does not have any real solutions because we cannot evaluate $\sqrt{-16}$ using real numbers.

As $\sqrt{-1}$ cannot be evaluated, mathematicians introduced a new symbol to denote it: i .

Thus, we have:

$$\begin{aligned} \sqrt{-1} &= i \\ -1 &= i^2. \end{aligned}$$

With this new symbol in hand, it follows that:

$$\sqrt{-16} = \sqrt{16 \times (-1)} = \sqrt{16} \times \sqrt{-1} = 4 \times \sqrt{-1} = 4i.$$

Here, $4i$ is an *imaginary number*.

An imaginary number is a number in the form bi , where b is a real number and $i = \sqrt{-1}$.

Now that we have introduced *imaginary numbers*, we can complete the solution for x in Example 1.

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{-16}}{2} \\ &= -3 \pm \frac{4i}{2} \\ &= -3 \pm 2i. \end{aligned}$$

We have just seen that by combining real numbers and imaginary numbers, we can find the solution to any quadratic equation.

3.2 The general complex number

A complex number is simply a number in the form $z = a + bi$, where a and b are real numbers, and $i = \sqrt{-1}$. a is called the *real* part of z , and b is called the *imaginary* part of z . We often use the following notations to denote the real and imaginary parts of z respectively :

$$a = \mathbf{Re}(z) \qquad b = \mathbf{Im}(z)$$

- If $\mathbf{Re}(z) = 0$, then $z = bi$, an imaginary number.
- If $\mathbf{Im}(z) = 0$, then $z = a$, a real number.

3.3 Complex arithmetic

Complex numbers can be added, subtracted and multiplied, just like real numbers.

3.3.1 Addition and subtraction

When adding or subtracting with complex numbers, we add their real and imaginary components separately.

Example 2: Let $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$. Find $z_1 + z_2$ and $z_1 - z_2$.

$$\begin{aligned} z_1 + z_2 &= (3 + 4i) + (2 + 3i) \\ &= 3 + 4i + 2 + 3i && \text{(Rearranging this)} \\ &= (3 + 2) + (4 + 3)i \\ &= 5 + 7i. \end{aligned}$$

Similarly,

$$\begin{aligned} z_1 - z_2 &= (3 + 4i) - (2 + 3i) \\ &= 3 + 4i - 2 - 3i && \text{(Rearranging this)} \\ &= (3 - 2) + (4 - 3)i \\ &= 1 + i \end{aligned}$$

3.3.2 Multiplication

In order to multiply complex numbers, we have to keep in mind that $i^2 = -1$. Once we have expanded the brackets, we only have to “collect” the real and imaginary terms before presenting our solution.

Example 3: Using the values of z_1 and z_2 from the previous example, calculate the value of $z_1 \cdot z_2$.

$$\begin{aligned} z_1 \cdot z_2 &= (3 + 4i) \cdot (2 + 3i) \\ &= 6 + 9i + 8i + 12i^2 && \text{(Since } i^2 = -1) \\ &= 6 + 9i + 8i + 12(-1) && \text{(Rearranging this)} \\ &= (6 - 12) + (9 + 8)i \\ &= -6 + 17i, \end{aligned}$$

Generally:

If $z_1 = a + bi$ and $z_2 = c + di$, then:

- $z_1 + z_2 = (a + c) + (b + d)i$
- $z_1 - z_2 = (a - c) + (b - d)i$
- $z_1 \cdot z_2 = (ac - bd) + (ad + bc)i$

3.4 The conjugate of a complex number

The *complex conjugate* of $z = a + bi$ is the number $a - bi$. We use a bar to denote the conjugate of a complex number.

Thus, the conjugate of $z = 3 + 7i$ is $\bar{z} = \overline{3 + 7i} = 3 - 7i$.

The one property of the complex conjugate that we will be looking at is if $z = a + bi$ and $\bar{z} = a - bi$, then the product of $z \cdot \bar{z}$ is a real number.

Proof:

$$\begin{aligned} z \cdot \bar{z} &= (a + bi) \cdot (a - bi) \\ &= a^2 - abi + abi - b^2i \\ &= a^2 + b^2 \end{aligned}$$

Therefore, when a complex number is multiplied by its conjugate, the imaginary parts are cancelled, and the product is a real number.

We will see in the following section that the division of complex numbers is slightly more complicated than multiplication, and usually we will have to make use of complex conjugates.

3.4.1 Division

Example 4: Let $z_1 = 3 + 4i$ and $z_2 = 2 + 3i$. Find $\frac{z_1}{z_2}$.

$$\frac{z_1}{z_2} = \frac{3 + 4i}{2 + 3i}$$

In order to make the denominator a real number, we have to multiply both the numerator and the denominator by $\bar{z}_2$.

Hence, we have :

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(3 + 4i)(2 - 3i)}{(2 + 3i)(2 - 3i)} \\ &= \frac{18 - i}{2^2 + 3^2} \\ &= \frac{18 - i}{13} \\ &= \frac{18}{13} - \frac{1}{13}i \end{aligned}$$

4 Written assignment

Questions marked with a “ * ” will involve concepts beyond the standard A-level syllabus and we strongly recommend reading through the notes in this booklet carefully before attempting these questions. Remember there may be one or two surprises along the way.

Question 1

Find the values of x which satisfy the following relationship:

$$\frac{2x+4}{x-1} = \frac{x+2}{x-\frac{1}{2}}.$$

[4 marks]

Question 2

Calculate the gradient at the point $x = \frac{1}{4}$ for the function $f(x) = \frac{4}{\sqrt{x}}$.

[5 marks]

Question 3

Given that $x = a^n$ and $y = a^m$ show that $\log_a(x) + \log_a(y) = \log_a(xy)$.

[5 marks]

Question 4

Show that $\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$

(Hint: draw and label a right angled triangle).

[6 marks]

Question 5*

The following question involves modular arithmetic where $[X]_n$ denotes $X \pmod{n}$. Determine answers to the following expressions:

- (a) $[47]_3$
- (b) $[-23]_5$
- (c) $[15 + 30 + 45 + \dots + 300]_5$
- (d) $[2 \times 4 \times 6 \times \dots \times 16]_3$

[4 marks]

Question 6*

For a given function $f(x) = x^2 + 4x + 5$;

- (a) Sketch the function and label any turning points and points of intersection with the axes.
- (b) Find the roots of $f(x)$.

[6 marks]

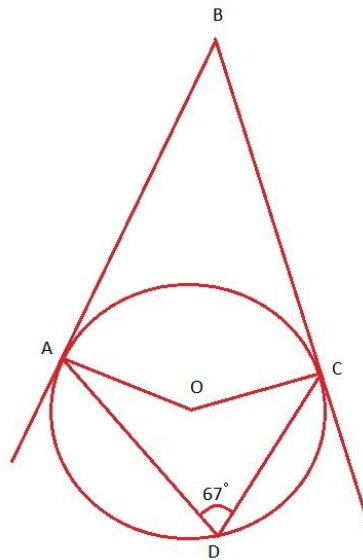
Question 7

Figure 1

Find the internal angle at point B of Figure 1.

[5 marks]

Question 8

A goat is tied to the middle of one side of a 10m square barn, using a 10m long rope. The barn is in the middle of a field. What is the area of grass the goat can graze?

[5 marks]

5 Research question

The final part of the project is a more open-ended, research-style question. We expect a one to two page report presenting all the information related to the research question.

Choose **one** of the following research topics.

[30 marks]

Geometry of complex numbers

We have shown you a brief introduction to the world of complex numbers. Investigate the geometry of complex numbers. How can they be drawn? What is their argument and modulus? Include any other information you think could be relevant or interesting.

The hyperbolic cosine function

You should all be familiar with the trigonometric function $\cos(\theta)$. Cosine has a *hyperbolic* cousin denoted $\cosh(\theta)$ (pronounced “cosh”). Investigate this function. How is it defined? What does its graph look like? What similarities and differences does it have to the normal cosine function? Also include any other information you feel is relevant or interesting.

Analysing summer project scores

The summer project marks from 10 students at two different schools are given in the tables below

Table 1

St John’s High										
Student number	1	2	3	4	5	6	7	8	9	10
Summer project percentage	74	91	67	51	61	84	76	60	63	59

Table 2

St Mary’s College										
Student number	1	2	3	4	5	6	7	8	9	10
Summer project percentage	59	63	96	73	41	71	63	53	59	86

First, analyse both datasets individually. What could an appropriate distribution be to fit to each set? Justify your answer. Estimate the parameter(s) in your chosen distribution and sketch their *probability density functions*. Find the 5th and the 95th percentiles for each distribution. Compare the two groups. Is there any evidence to suggest that the marks from the two groups are statistically different? Justify all of your conclusions.

Climate temperature data

Find some data related to global temperature over a time scale of your choosing. Comment on the validity of the data. Use appropriate statistical methods to analyse your data. Identify any important summaries (eg. median, standard deviation). Is there any evidence to suggest the climate is changing? If so, comment on the reliability of any conclusions you make. You are only expected to analyse a dataset containing a maximum 20 observations. If you find a dataset containing more than 20 observations during your research then just use a representative sub-set.