Modelling spatial dependence in rainfall extremes

Lee Fawcett, Dave Walshaw and Simone Padoan

ASP seminar — 20th June 2013

Structure of this talk

1. Background and motivation
2. Models for spatial data
3. Limiting models for extremes
4. Application
5. Simulation study

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1. Background and motivation

In the last fifteen years in the UK there has been a notable increase in the **frequency** – and **severity** – of extreme precipitation events.

In many instances, flooding has occurred as a result of rivers bursting their banks – which itself has often been the result of the observation of **simultaneous** rainfall extremes at a number of sites within the catchment.

1. Background and motivation

Models from **extreme value theory** that have been used to characterise the spatial nature of such rainfall extremes are restrictive:

- Assumption of **max–stability** – dependence holds for all events more extreme than those that have already happened
- In reality – dependence may be observed for data above levels of practical interest, but these data may be independent in the limit...
- ... leading to **non–convergence** of the dependence structure

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The year 2012 was the second wettest year in the UK since records began, and six years since 1998 are in the **top ten wettest years** (1998, 1999, 2000, 2002, 2008 and 2012).

Flooding (usually in the summer) now seems to be an accepted part of the British climate!

1.1 Examples

The year 2012 was the second wettest year in the UK since records began, and six years since 1998 are in the **top ten wettest years** (1998, 1999, 2000, 2002, 2008 and 2012).

Flooding (usually in the summer) now seems to be an accepted part of the British climate!

1.1 Examples: Flooding in Boscastle (2004)

- Over 100 homes/businesses destroyed
- Two bridges destroyed
- No loss of life

1.1 Examples: Flooding in Tewkesbury (2007, 2012)

- City completely cut off
- 2,100 callouts in two days to Gloucestershire Fire & Rescue – normally 8,000 per year
- Water supply badly affected

1.1 Examples: Flooding in Cumbria (2009)

- £100 million worth of damage
- A number of deaths
- Massive transport disruption

1.1 Examples: Flash–flooding in Newcastle (2012)

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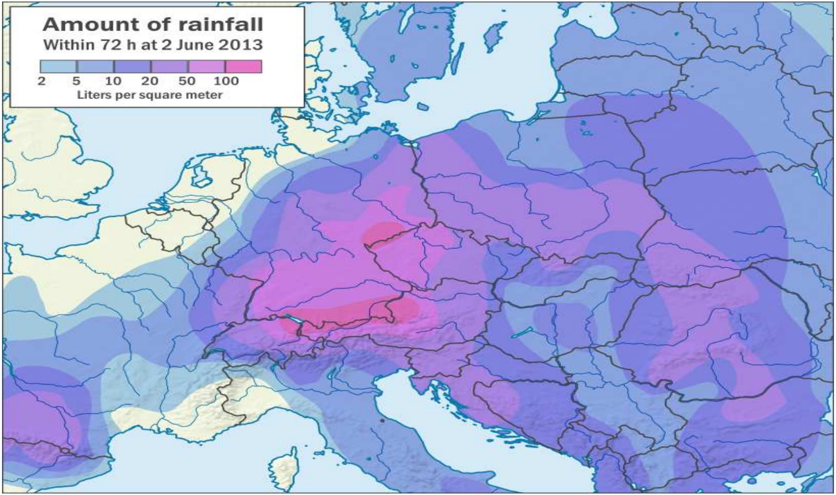

1.2 Useful parameters?

The 2009 floods in Cumbria occurred as a result of rivers (Derwent & Cocker) bursting their banks – rainfall extremes were observed at many locations **simultaneously**.

Thus, of particular interest here might be the estimation of:

(i) $\mathcal{P}_r$ – joint exceedance probabilities of the marginal r–year quantiles, at a collection of sites within a region

(ii) $\mathcal{T}_r$ – the total maximum rainfall accumulation we would expect to see (on average) once every r years within a region

Such quantities are both **multivariate** and **spatial** in nature.

Data

- Daily rainfall totals (mm) for 206 sites in the UK, 1961 $\longrightarrow$ 2000 (inclusive)
- Nine geographically–defined climate regions – areas of different physiological character (Wigley *et al.*, 1984)
- To focus on extremes, we filter out a set of 40 **annual maxima** for each site
- Within each region, a single multivariate **observation** is the set of annual maxima from sites within that region (a single **replication** in our simulation study)

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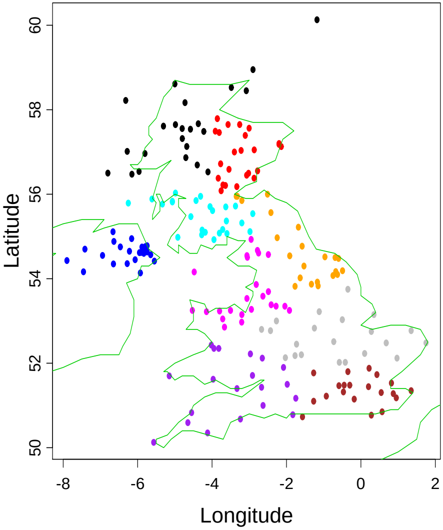
1.3 Our work



Aims

- Investigate the consequences of **model mis–specification** on the estimation of quantities such as $\mathcal{P}_r$ and $\mathcal{T}_r$
- How do estimates of $\mathcal{P}_r$ and $\mathcal{T}_r$ depend on our assumptions relating to spatial dependence in the extremes?
- If there is **genuine extremal dependence**, models which give asymptotic independence might under–estimate $\mathcal{P}_r$ and $\mathcal{T}_r$. *But by how much, and is this significant?*
- If we have only **sub–asymptotic dependence**, using max–stable models might over–estimate $\mathcal{P}_r$ and $\mathcal{T}_r$. *But by how much, and is this significant?*
- For real data, can we tell whether or not any spatial dependence present is asymptotic?

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By far the most simple and well–understood approach to modelling spatial data is to assume our spatial process $\{Y(x)\}$ is **Gaussian**, that is

$$\{Y(x)\} \sim \mathbb{GP}(\mu, \rho),$$

where μ and ρ are the **mean** and **correlation functions** (respectively).

- $\{Y(x)\}$ is a stochastic process
- Any finite linear combination of samples has a joint Gaussian distribution

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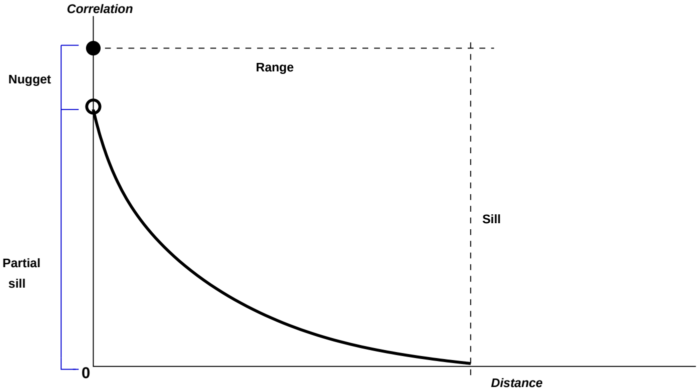
Anatomy of a typical correlation function:

[Cressie (1993), Wackernagel (2003), Diggle & Ribeiro (2007)]



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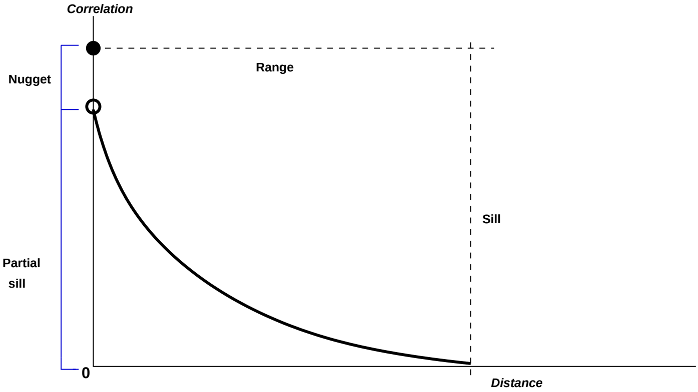

Standard correlation functions:

Model	Parameters (scale λ, shape ν)	Correlation function (distance h)
Exponential	$\lambda > 0$	$\exp\{-h/\lambda\}$
Powered exponential	$\lambda > 0$ $0 < \nu \leq 2$	$\exp\{-(h/\lambda)^{\nu}\}$
Whittle–Matérn	$\lambda > 0$ $\nu > 0$	$\dfrac{(h/\lambda)^{\nu}}{2^{\nu-1}\Gamma(\nu)} L_{\nu}(h/\lambda)$

- **Isotropic**
- **Anisotropy** can be added by replacing h with $(h^{\mathsf{T}}Ah)^{0.5}$, where A is a positive definite matrix with determinant 1
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2. Models for spatial data: Gaussian processes

- Gaussian–based geostatistical models are **widely–understood**
- Extremely **flexible** owing to the range of correlation functions that are available
- Idea: Transform annual maxima to Gaussian $\longrightarrow$ use standard methods from the geostatistical toolbox to model dependence structure
- Problem #1: The model for the original data should be max–stable $\longrightarrow$ restricts structure of the correlation function that can be used
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Standard Gaussian–based geostatistical models might be inappropriate for our rainfall annual maxima.

The standard models for extremes have been extended to provide a spatial interpretation: **max–stable processes**.

Suppose $Y_1, Y_2, \ldots, Y_n$ are independent and identically distributed random variables with distribution function F.

Then, if there exists sequences of constants $\{a_n\} > 0$ and $\{b_n\}$ such that

$$\Pr\left\{a_n^{-1}\left(\max\left[Y_1, Y_2, \ldots, Y_n\right] - b_n\right) \leq x\right\} \to G(x)$$

as $n \to \infty$, G must be **max–stable**; that is,

$$G^n(b'_n + a'_n x) = G(x)$$

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3.1 Univariate considerations

The only non–degenerate distribution with this property is the **generalised extreme value** (GEV) distribution:

$$\mathcal{G}(x;\mu,\sigma,\xi) \quad = \quad \begin{cases} \exp\left[-\left(1+\xi(x-\mu)/\sigma\right)_+^{-1/\xi}\right], & \xi \neq 0 \\ \exp\left[-\exp\left(-(x-\mu)/\sigma\right)\right], & \xi = 0, \end{cases}$$

where $-\infty < \mu < \infty$, $\sigma > 0$ and $-\infty < \xi < \infty$ are location, scale and shape parameters (respectively).

Estimates of an extreme quantile z_r can be obtained by inversion of the above, giving the r–year **return level**:

$$\hat{z}_r \quad = \quad \begin{cases} \mu + \dfrac{\sigma}{\xi}\left[\left(-\log\left(1-r^{-1}\right)\right)^{-\xi} - 1\right], & \xi \neq 0, \\ \mu - \sigma\log\left[-\log\left(1-r^{-1}\right)\right], & \xi = 0. \end{cases}$$

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3.2 Extension to the multivariate setting

Now suppose we have independent sequences of D–variate random variables $(Y_{1i}, Y_{2i}, \ldots, Y_{Di})$, $i = 1, 2, \ldots, n$, with **componentwise maxima**

$$\max[Y_{11}, Y_{12}, \ldots, Y_{1n}], \ldots, \max[Y_{D1}, Y_{D2}, \ldots, Y_{Dn}].$$

If it exists (and is non–degenerate), the limiting joint distribution of these maxima (after marginal transformation to standard Fréchet), for $z > 0$, can be written as

$$\Pr\{Z_1 \le z_1, Z_2 \le z_2, \ldots, Z_D \le z_D\} = \exp\{-V(z_1, z_2, \ldots, z_D)\},$$

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$$\Pr\{Z_1 \leq z_1, Z_2 \leq z_2, \ldots, Z_D \leq z_D\} = \exp\{-V(z_1, z_2, \ldots, z_D)\},$$

with $z_1, z_2, \ldots, z_D > 0$.

The infinite–dimensional analogues of multivariate extreme value distributions are **max–stable processes** (de Haan, 1984).

More precisely, consider a stochastic process $\{Y(x)\}$, $x \in \mathbb{R}^D$, having continuous sample paths. Then the limiting process is given by

$$\max\left\{a_n^{-1}(x)\left(\max\left[Y_1(x), Y_2(x), \ldots, Y_n(x)\right] - b_n(x)\right)\right\} \to \{Z(x)\},$$

as $n \to \infty$, where

- Y_i, $i = 1, \ldots, n$ are independent replications of Y
- $a_n(x) > 0$ and $b_n(x) \in \mathbb{R}^D$ are sequences of continuous functions
- The limiting process Z is assumed to be non–degenerate

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3.3 Extending further: the infinite–dimensional case

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Schlather (2002) shows that $Z(x)$ can be written, according to its spectral characterisation, as:

$$Z(x) \quad = \quad \max\left[\zeta_1 Y_1(x), \zeta_2 Y_2(x), \ldots\right],$$

where $\zeta_1, \zeta_2, \ldots$ are the points of a Poisson process on $(0, \infty]$ with intensity $d\Lambda(\zeta) = \zeta^{-2} d\zeta$ and $\mathbb{E}[Y(x)] = 1$ for all $x \in \mathbb{R}^D$.

Different choices for the process Y give some useful max–stable processes:

- Gaussian extreme value process (Smith,1990)
- Extremal Gauss process (Schlather, 2002)
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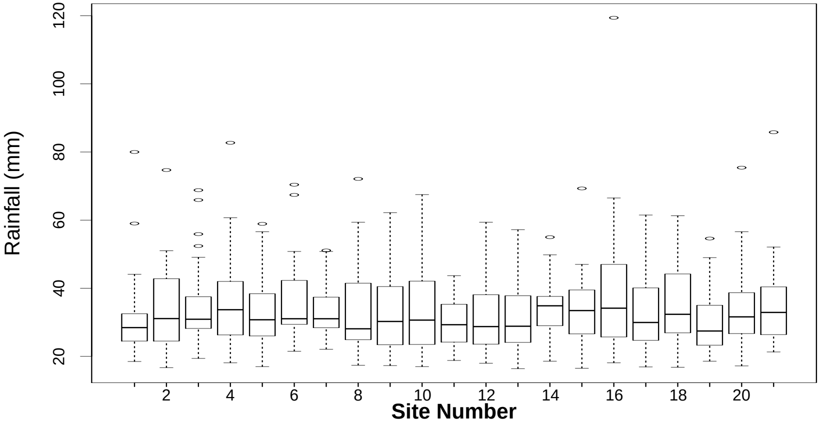
4. Application to rainfall in Central/Eastern England



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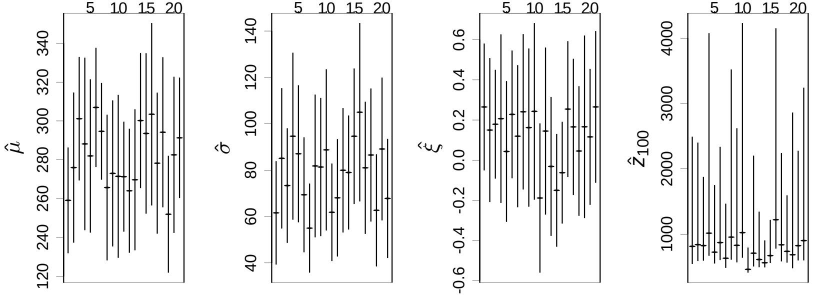

4. Application to rainfall in Central/Eastern England

- Transform margins to standard normal using

$$\Phi^{-1}\{\mathcal{G}_j(x; \hat{\mu}_j, \hat{\sigma}_j, \hat{\xi}_j)\};$$

then fit a simple Gaussian process

- Transform margins to standard Fréchet using

$$-\log\{\mathcal{G}_j(x; \hat{\mu}_j, \hat{\sigma}_j, \hat{\xi}_j)\}^{-1};$$

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Suppose our data matrix for CEE is:

$$\mathcal{X} = \begin{array}{c} \\ 1961 \\ 1962 \\ \vdots \\ 2000 \end{array} \begin{array}{c} \text{site 1} \quad \text{site 2} \quad \cdots \quad \text{site 21} \\ \left(\begin{array}{cccc} \mathcal{X}_1 & \longrightarrow & & \\ \mathcal{X}_2 & \longrightarrow & & \\ \vdots & \vdots & & \\ \mathcal{X}_{40} & \longrightarrow & & \end{array} \right) \end{array}$$

If the parameters of the model ϑ can be identified from the pairwise marginal densities, then we maximise a **composite log–likelihood** of the form

$$\ell_p(\vartheta) = \sum_{i=1}^{40} \sum_{\{j<k;x_j,x_k \in \mathcal{X}_i\}} \log f(x_j, x_k; \vartheta).$$

The variance matrix is found via an information sandwich:

$$\mathrm{var}(\vartheta) = J^{-1}(\hat{\vartheta})K(\hat{\vartheta})J^{-1}(\hat{\vartheta}).$$

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4. Application to rainfall in Central/Eastern England

Model	Correlation	Scale (λ)	Shape (ν)	$\ell_{\max}$	CLIC
Gaussian process	Exponential	56.444	—	–706.52	—
		(44.937)			
	Powered exponential	44.390	0.689	–705.45	—
		(141.503)	(1.607)		
	Whittle–Matérn	78.569	0.328	–706.18	—
		(220.167)	(1.785)		
Extremal Gauss process	Exponential	168.025	—	–23,575	47,034
		(64.105)			
	Powered exponential	168.344	1.075	–23,572	47,020
		(64.724)	(0.188)		
	Whittle–Matérn	168.424	0.158	–23,572	47,010

5. Simulation study

- Simulate a **master** random field **M** (Gaussian/max–stable):

$$
\mathbf{M} = \begin{array}{c} \\ \text{rep. } 1 \\ \text{rep. } 2 \\ \text{rep. } 3 \\ \vdots \\ \text{rep. } N \end{array}
\begin{array}{c}
\begin{array}{cccc} \text{site } 1 & \text{site } 2 & \cdots & \text{site } n \end{array} \\
\begin{pmatrix}
(\mu,\sigma,\xi) & & & \\
\text{margins} & (\lambda,\nu) & & \\
 & \text{spatial} & & \\
 & & U(0,10) \text{ co-ord.} & \\
 & & &
\end{pmatrix}
\end{array}
$$

- Fit *correct* spatial process to **M**: $(\hat{\lambda},\hat{\nu})$ (+ others: sill, range...)
- Fit *incorrect* spatial process to **M**: $(\hat{\lambda},\hat{\nu})^{☠}$ (+ others: sill, range...)
- Simulate K replications using $(\hat{\lambda},\hat{\nu})$ and $(\hat{\lambda},\hat{\nu})^{☠}$, giving

$$
\begin{array}{c} \\ \text{rep. } 1 \\ \vdots \\ \text{rep. } K \end{array}
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\begin{pmatrix} & & & \\ & & & \\ & & & \end{pmatrix}
\end{array}
\qquad
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\begin{pmatrix} & & & \\ & & ☠ & \\ & & & \end{pmatrix}
\end{array}
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- We use $(\mu,\sigma,\xi) = (300, 80, 0.1)$; $(n, N, K) = (50, 100, 10^6)$; $\lambda = (1, 20, 50)$ and $\nu = (0.5, 1, 1.5, 2)$.

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- Simulate K replications using $(\hat{\lambda}, \hat{\nu})$ and $(\hat{\lambda}, \hat{\nu})^{\skull}$, giving

$$\begin{array}{cc} & \begin{array}{cccc}\text{site 1} & \text{site 2} & \cdots & \text{site } n\end{array} \\ \begin{array}{c}\text{rep. 1}\\ \vdots\\ \text{rep. } K\end{array} & \left(\begin{array}{c} \\ \\ \\ \end{array}\right)\end{array} \qquad \begin{array}{cc} & \begin{array}{cccc}\text{site 1} & \text{site 2} & \cdots & \text{site } n\end{array} \\ \begin{array}{c}\text{rep. 1}\\ \vdots\\ \text{rep. } K\end{array} & \left(\begin{array}{c} \\ \skull \\ \\ \end{array}\right)\end{array}$$

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5.1 Quantities of interest: $\mathcal{P}_r$

- Let $\mathcal{P}_r$: joint exceedance probabilities of the marginal r-year return levels z_r for S 'sites of interest' in our 'region'

- We use $S = 3$ and $S = 10$: choosing randomly, we get
  - $s = \{23, 43, 47\}$ for $S = 3$
  - $s = \{41, 18, 12, 13, 8, 4, 32, 5, 17, 47\}$ for $S = 10$

- Then, for each of the correct and incorrect models, we get:

$$\widehat{\mathcal{P}}_r = \sum_{k=1}^{K} \mathrm{I}_k / K,$$

where

$$\mathrm{I}_k = \begin{cases} 1 & \text{if } y_{k,s_t} > z_r, \qquad t = 1, \ldots, S; \\ 0 & \text{otherwise,} \end{cases}$$

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- We use $S = 3$ and $S = 10$: choosing randomly, we get
  - $s = \{23, 43, 47\}$ for $S = 3$
  - $s = \{41, 18, 12, 13, 8, 4, 32, 5, 17, 47\}$ for $S = 10$

- Then, for each of the correct and incorrect models, we get:

$$\widehat{\mathcal{P}}_r = \sum_{k=1}^{K} \mathrm{I}_k / K,$$

where

$$\mathrm{I}_k = \begin{cases} 1 & \text{if } y_{k,s_t} > z_r, \qquad t = 1, \ldots, S; \\ 0 & \text{otherwise,} \end{cases}$$

at each replication $k = 1, \ldots, K$.

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5.1 Quantities of interest: $\mathcal{T}_r$

Recall that $\mathcal{T}_r$ is the total maximum rainfall accumulation we would expect to see once, every r year, across an entire region.

Difficult to estimate, since we have a finite number of ‘sites’.

- For each replication k in our simulated data frame, $k = 1, \ldots, K$, we find:

$$T_k = \sum_{i=1}^{n} y_{k,i}.$$

- Denoting by $T^{[k]}$ the k–th order statistic of $\{T_1, T_2, \ldots, T_K\}$, we then estimate empirically the r–year return level for aggregate values across all replications as

$$\hat{z}_{r(\mathrm{agg})} \;=\; T^{[K(1-r^{-1})]}.$$

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5.2 Sampling error

Our estimates $\widehat{\mathcal{P}}_r$ and $\hat{z}_{r\,(\mathrm{agg})}$ are one–off estimates obtained empirically from the simulated data.

To assess the variability of these estimates, we *could* repeat the simulation procedure multiple times, each time with $K = 10^6$.

However, computationally this would be rather burdensome!

Rather, we use **bootstrap methods** to estimate $\mathrm{var}[\widehat{\mathcal{P}}_r]$, $\mathrm{var}[\hat{z}_{r\,(\mathrm{agg})}]$ and confidence intervals for both.

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5.2 Sampling error: Bootstrap scheme

- For each bootstrap replication b, $b = 1, \ldots, B$, we randomly sample (with replacement) K rows from each $(K \times n)$ matrix of simulated data

- For each bootstrap replication of the simulated spatial process, we find $\widehat{\mathcal{P}}_r$ and $\hat{z}_{r(\mathrm{agg})}$, yielding a collection of estimates

$$\left\{\widehat{\mathcal{P}}_r^{(1)}, \hat{z}_{r(\mathrm{agg})}^{(1)}, \ldots, \widehat{\mathcal{P}}_r^{(B)}, \hat{z}_{r(\mathrm{agg})}^{(B)}\right\},$$

  from which we can estimate variances or construct confidence intervals

- Using $B = 1000$, we obtain **bias–corrected accelerated** (BC$_a$) intervals as proposed in Efron (1987), making use of a jackknife procedure

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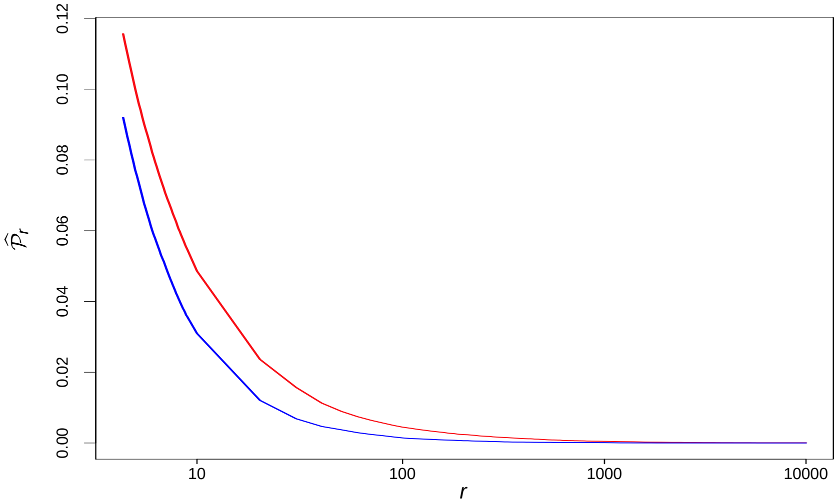
5.3 Some results

True spatial process: Extremal Gauss ($n = 50$, $N = 100$) *Correlation model:* Exponential, $\lambda = 20$		*Simulated process* ($n = 50$, $K = 10^6$) Extremal Gauss ($\hat{\lambda} = 21.043$)		Standard Gaussian process ($\hat{\lambda} = 117.367$)	
$\widehat{\mathcal{P}}_r$ $\times 100\%$ ($S = 3$)	$r = 10$	5.141	(5.000, 5.284)	3.303	(3.184, 3.409)
	$r = 50$	1.040	(0.978, 1.105)	0.367	(0.332, 0.403)
	$r = 200$	0.254	(0.223, 0.284)	0.068	(0.052, 0.084)
	$r = 1000$	0.041	(0.030, 0.055)	0.010	(0.004, 0.017)
	$r = 10000$	0.005	(0.001, 0.010)	0.002	(0.000, 0.005)
$\widehat{\mathcal{P}}_r$ $\times 100\%$ ($S = 10$)	$r = 10$	3.227	(3.111, 3.336)	1.531	(1.456, 1.607)
	$r = 50$	0.632	(0.581, 0.684)	0.119	(0.097, 0.140)
	$r = 200$	0.136	(0.114, 0.161)	0.014	(0.007, 0.022)
	$r = 1000$	0.029	(0.019, 0.041)	0.000	—
	$r = 10000$	0.002	(0.000, 0.005)	0.000	—
$\hat{z}_{r\,(\mathrm{agg})}$ (thousand)	$r = 10$	24.182	(24.096, 24.267)	24.336	(24.254, 24.423)
	$r = 50$	32.299	(32.051, 32.520)	31.763	(31.570, 31.973)
	$r = 200$	40.308	(39.745, 40.966)	38.738	(38.331, 39.123)
	$r = 1000$	50.948	(49.826, 51.972)	47.651	(46.452, 49.302)
	$r = 10000$	71.626	(66.681, 75.694)	63.732	(59.630, 71.285)

5.3 Some results



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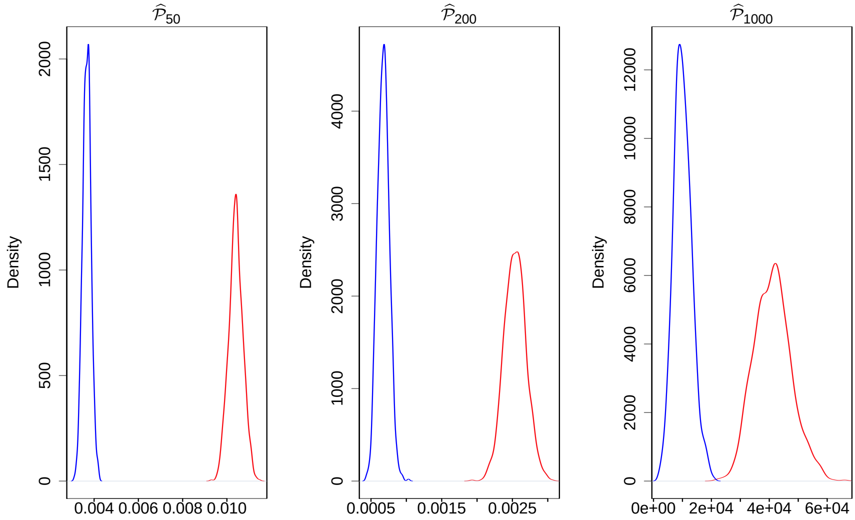

Further work

- Diagnostic checks for max–stability in UK rainfall extremes
- Comparison of regional estimates of $\mathcal{P}_r$ and $z_{r\,(\mathrm{agg})}$ for UK data
- Mixture modelling to account for asymptotic dependence/independence between rainfall extremes for sites within the same region

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