Lecture 10

LINEAR PROGRAMMING (II)

Announcements

CBA6

- Has gone "live" in practice mode this morning
- Covers all material in Chapters 6, 7 8 and 9 (today)

No lecture next Monday – Bank Holiday

Tutorials next week as normal – will discuss specimen exam paper and start revision

Graphical solutions for two–variable problems

In last week's lecture we discussed how to **formulate** a linear programming problem.

This week, we consider how to **solve** such problems.

Linear programming problems that involve only two decision variables, x and y, may be solved **graphically** – that is, by drawing lines associated with the constraints, identifying a region of possible solutions and then locating a point in this region that has a particular property.

We will now re–visit the examples introduced in last week's lecture.

Example (A chair manufacturer)

Recall the problem posed in **example 8.2.1** (last week):

A manufacturer makes two kinds of chairs, A and B, each of which has to be processed in two departments, I and II. Chair A has to be processed in department I for 3 hours and in department II for 2 hours. Chair B has to be processed in department I for 3 hours and in department II for 4 hours.

The time available in department I in any given month is 120 hours, and the time available in department II, in the same month, is 150 hours.

Chair A has a selling price of £10 and chair B has a selling price of £12.

The manufacturer wishes to maximise his income. How many of each chair should be made in order to achieve this objective? You may assume that all chairs made can be sold.

Example (A chair manufacturer)

The table below summarises the information given in this particular problem:

Type of chair	**Time dept. I**	**Time dept. II**	**Price (£)**
A	3	2	10
B	3	4	12
Total time available	**120**	**150**	

Example (A chair manufacturer)

Recall the three steps involved in formulating a linear programming problem:

1. Identify the **decision variables**
2. Identify the **constraints**
3. State the **objective function**

In this example, the decision variables were identified as

$$\begin{aligned} x &= \text{number of type A chairs made} \qquad \text{and} \\ y &= \text{number of type B chairs made.} \end{aligned}$$

Example (A chair manufacturer)

The constraints were

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

We then identified that the objective was to maximise the income, which we called Z, where

$$Z = 10x + 12y.$$

Example (A chair manufacturer)

So, in summary, the linear programming problem is:

Maximise $Z = 10x + 12y$ subject to the following constraints:

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

Example (A chair manufacturer)

To find the **feasible region** for this problem (i.e. the region on a graph which satisfies all of our inequalities), we proceed by indicating, on a diagram, the region for which all of the inequalities hold.

Example (A chair manufacturer)

The first inequality is $3x + 3y \leq 120$. To show this on a graph, we first need to plot the line $3x + 3y = 120$.

- When $x = 0$, we have

$$\begin{aligned} 3 \times 0 + 3y &= 120 \qquad \text{i.e.} \\ 3y &= 120 \qquad \text{i.e.} \\ y &= 40. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 3x + 3 \times 0 &= 120 \qquad \text{i.e.} \\ 3x &= 120 \qquad \text{i.e.} \\ x &= 40. \end{aligned}$$

Example (A chair manufacturer)

Now consider the second inequality $2x + 4y \leq 150$. Again, to show this on a diagram, we first need to plot the line $2x + 4y = 150$.

- When $x = 0$, we have

$$\begin{aligned} 2 \times 0 + 4y &= 150 \qquad \text{i.e.} \\ 4y &= 150 \qquad \text{i.e.} \\ y &= 37.5. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 2x + 4 \times 0 &= 150 \qquad \text{i.e.} \\ 2x &= 150 \qquad \text{i.e.} \\ x &= 75. \end{aligned}$$

On figure 9.1 we also shade out the inadmissible regions for the two non–negativity constraints.

Example (A chair manufacturer)

The unshaded region in figure 9.1 shows the feasible region associated with our set of inequalities.

What we must do now is find the point in that region which meets our objective – i.e. the point in that region which **maximises income**.

One way of doing this is to also plot the **objective function**.

Example (A chair manufacturer)

Now our objective function is

$$Z = 10x + 12y,$$

where Z is our income.

When Z takes different values we get a family of **parallel straight lines**.

We need to choose a starting value for Z in order to be able to plot the objective function.

Example (A chair manufacturer)

It's often a good idea to try a value which is a multiple of both the coefficients of x and y.

The coefficient of x is 10 and the coefficient of y is 12, so we could try a starting value of $Z = 120$. Thus, the objective function is now

$$10x + 12y = 120.$$

We can plot this line in the same way as before – i.e. consider what happens when x and y are zero.

Example (A chair manufacturer)

- When $x = 0$, we have

$$\begin{aligned} 10 \times 0 + 12y &= 120 \qquad \text{i.e.} \\ 12y &= 120 \qquad \text{i.e.} \\ y &= 10. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 10x + 12 \times 0 &= 120 \qquad \text{i.e.} \\ 10x &= 120 \qquad \text{i.e.} \\ x &= 12. \end{aligned}$$

This line should also plotted on figure 9.1.

- Notice that this line does not give the **optimal income**
- The origin represents zero income, and we want to move as far away from this as possible.

Example (A chair manufacturer)

You should see from your graph that the point within th feasible region that is furthest away from the origin, **but parallel to the objective function**, is $\mathbf{x = 5}$ and $\mathbf{y = 35}$.

Thus, to maximise income, we should make **5 type A chairs** and **35 type B chairs**.

This will give an income of

$$\begin{aligned} Z &= 10x + 12y \qquad \text{i.e.} \\ Z &= 10 \times 5 + 12 \times 35 \qquad \text{i.e.} \\ Z &= 50 + 420 \\ &= 470, \end{aligned}$$

i.e. £470.

Example (Replica sports shirts)

Recall the example about *Sportizus*, a company who produce replica football and rugby shirts.

	Sewing (mins)	Transfer (minutes)	Profit (P)
Football	**8**	**9**	**22**
Rugby	**5**	**15**	**16**
Total time	**600**	**900**	

Example (Replica sports shirts)

Last week, we summarised this problem as follows:

Maximise $P = 22x + 16y$ subject to the following constraints:

$$\begin{aligned} 8x + 5y &\leq 600, \\ 9x + 15y &\leq 900, \\ x &\geq 30 \qquad \text{and} \\ y &\geq 10, \end{aligned}$$

where x and y are the number of football and rugby shirts to make, respectively.

Example (Replica sports shirts)

The first inequality is $8x + 5y \leq 600$. First, let's plot the line $8x + 5y = 600$.

- When $x = 0$, we have

$$\begin{aligned} 8 \times 0 + 5y &= 600 \qquad \text{i.e.} \\ 5y &= 600 \qquad \text{i.e.} \\ y &= 120. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 8x + 5 \times 0 &= 600 \qquad \text{i.e.} \\ 8x &= 600 \qquad \text{i.e.} \\ x &= 75. \end{aligned}$$

Example (Replica sports shirts)

The second inequality is $9x + 15y \leq 900$. First, let's plot the line $9x + 15y = 900$.

- When $x = 0$, we have

$$\begin{aligned} 9 \times 0 + 15y &= 900 \qquad \text{i.e.} \\ 15y &= 900 \qquad \text{i.e.} \\ y &= 60. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 9x + 15 \times 0 &= 900 \qquad \text{i.e.} \\ 9x &= 900 \qquad \text{i.e.} \\ x &= 100. \end{aligned}$$

Example (Replica sports shirts)

We also require $x \geq 30$ and $y \geq 10$.

Thus, on figure 9.2, you should also shade out all x values less than 30 and all y values less than 10, to leave $x \geq 30$ and $y \geq 10$ *unshaded*.

Example (Replica sports shirts)

Finally, we need to plot the **objective function**. Here, we have

$$P = 22x + 16y;$$

as with the previous example, we need a starting value for P in order to plot this line.

You should notice that starting with $P = 22 \times 16 = 352$ will give $(0, 22)$ and $(16, 0)$, and so we also plot this line on figure 9.2.

Example (Replica sports shirts)

Notice that this objective line lies outside of the feasible region for this problem.

Let's see what happens when we try a bigger starting value for the profit P – for example, let's try a starting value for P four times as big – i.e. $P = 4 \times 352 = 1408$.

Example (Replica sports shirts)

- When $x = 0$,

$$\begin{aligned} 22 \times 0 + 16y &= 1408 \qquad \text{i.e.} \\ 16y &= 1408 \qquad \text{i.e.} \\ y &= 88. \end{aligned}$$

- When $y = 0$,

$$\begin{aligned} 22x + 16 \times 0 &= 1408 \qquad \text{i.e.} \\ 22x &= 1408 \qquad \text{i.e.} \\ x &= 64. \end{aligned}$$

Example (Replica sports shirts)

To maximise profit, $\mathbf{x = 60}$ and $\mathbf{y = 24}$, i.e. we should make **60 football shirts** and **24 rugby shirts** each day.

This will give a maximum daily profit of

$$\begin{aligned} P &= 22x + 16y \\ &= 22 \times 60 + 16 \times 24 \\ &= £1704. \end{aligned}$$

Algebraic solutions for two–variable problems

Notice that in both the examples we have looked at so far, the graphical representation indicates that the solution is at the intersection of two lines on the graph.

Thus, we could find also the solution **algebraically**, as the solution to a pair of **simultaneous equations**.

The chair manufacturer (revisited)

Notice that the solution to the chair manufacturer's problem lay at the intersection of the two lines

$$3x + 3y = 120 \qquad \text{and} \tag{1}$$
$$2x + 4y = 150. \tag{2}$$

We can solve this pair of simultaneous equations by **elimination**.

Notice that if we multiply equation (1) by 2 and multiply equation (2) by 3, we get

$$6x + 6y = 240 \qquad \text{and} \tag{3}$$
$$6x + 12y = 450. \tag{4}$$

Now the x's have the same **coefficient**.

The chair manufacturer (revisited)

Subtracting equation (3) from equation (4) gives

$$\begin{aligned} 0x + 6y &= 210 && \text{and so} \\ 6y &= 210 && \text{i.e.} \\ y &= 35. \end{aligned}$$

Substituting $y = 35$ into equation (1) then gives

$$\begin{aligned} 3x + 3 \times 35 &= 120 \\ 3x + 105 &= 120 \\ 3x &= 15 \\ x &= 5. \end{aligned}$$