Lecture 9

LINEAR PROGRAMMING (I)

Announcements

Assignment 2

- Due in this coming **Friday**, **1pm** (surnames N→Z), **4pm** (surnames A→M)
- Must hand your work in to the **Maths & Stats General Office**
- I will be available on Wednesday afternoon, 2–3pm, for any last minute queries

CBA6

- Will go "live" in practice mode next week
- All about the material in Chapters 6, 7 and 8

Specimen exam paper to go live soon

Introduction

Decision making is a process that is carried out in many areas of life.

Usually there is a particular aim in making one decision rather than another.

Two aims often considered in business are:

- **maximising profit**, and
- **minimising cost**.

During World War Two, American mathematicians developed some mathematical methods to help decision making processes.

Their aims were to express all

- **requirements**
- **constraints** and
- **objectives**

as algebraic equations. They then developed methods for obtaining the **optimal solution** to the problem posed.

One such method is called **linear programming**.

Linear programming belongs to a field of statistics known as **operational research**.

For our set of algebraic equations to reflect the **requirements**, **constraints** and **objectives** of a real–life situation, you can imagine how complex they would be!

In this chapter, we will study simple problems, for which all the algebraic expressions are **linear**, i.e.

$$\textbf{(a number)}x + \textbf{(a number)}y \quad = \quad \textbf{a number}.$$

For example, we might express **profit** as a linear combination of two other variables:

$$4x + 3y \quad = \quad \text{Profit}.$$

This is a linear **equation**.

If we want our profit to be *at least* £50, we might consider the following linear **inequality**:

$$4x + 3y \geq £50.$$

In today's lecture, we will consider how to **formulate** real–life situations as linear programming problems.

Next week, we will discuss how to **solve** such problems.

Formulating the problem

1. Identify the **decision variables**
   These are the quantities you need to know in order to solve the problem.
2. Identify the **constraints**
   For example, there may be a limit on resources or a maximum/minimum value a decision variable can take.
3. Determine the **objective function**
   This is the quantity to be be *optimised*, usually **profit** or **costs**.

Formulating the problem

We will consider three real–life examples:

- A **chair manufacturer**,
- A **producer of replica football/rugby shirts**, and
- A **haulage company**.

Example 1: A chair manufacturer

A manufacturer makes two kinds of chairs – **A** and **B**. Each type of chair has to be processed in two departments – $\mathbf{I}$ and $\mathbf{II}$.

Chair **A** spends 3 hours in department $\mathbf{I}$ and 2 hours in department $\mathbf{II}$. Chair **B** spends 3 hours in department $\mathbf{I}$ and 4 hours in department $\mathbf{II}$.

The time available in department $\mathbf{I}$ in any given month is 120 hours, and the time available in department $\mathbf{II}$ in the same month is 150 hours.

Chair **A** has a selling price of £10 and chair **B** of £12.

The manufacturer wishes to maximise his income.

How many of each type of chair should be made?

You'll notice that there's a lot of information given in the question – this is typical of a linear programming problem. Sometimes it's easier to summarise the information given in a table:

Chair	Time in dept. I	Time in dept. II	Selling price
A	3	2	10
B	3	4	12
Time limits	120	150	

To formulate this linear programming problem, we consider the following three steps:

1. What are the **decision variables**? (i.e. which quantities do you need to know in order to solve the problem?)
2. What are the **constraints**?
3. What is the **objective**?

Step 1: Decision variables

Read through the question and identify the things you'd like to know. You can usually do this by going straight to the last sentence of the question:

"How many of each chair should be made..."

Thus, we'd like to know

- the number of type **A** chairs to make, and
- the number of type **B** chairs to make.

These are our decision variables, and are usually denoted with lower case letters. Thus, our decision variables are

$$x = \text{number of type } \mathbf{A} \text{ chairs made} \qquad \text{and}$$
$$y = \text{number of type } \mathbf{B} \text{ chairs made.}$$

Step 2: Constraints

This is probably the hardest bit! Consider what could happen in each department.

For example, if we focus on what could happen in department $\mathbf{I}$:

Since:	the production of 1 type **A** chair uses 3 hours,
then:	the production of x type **A** chairs takes $3x$ hours.
Similarly:	the production of 1 type **B** chair uses 3 hours,
so:	the production of y type **B** chairs takes $3y$ hours.

The total time used in department $\mathbf{I}$ is therefore

$$(3x + 3y) \text{ hours.}$$

Since only 120 hours are available in department $\mathbf{I}$, one constraint is

$$\begin{aligned} (3x + 3y) \text{ hours} &\leq 120 \text{ hours}, \qquad \text{or just} \\ (3x + 3y) &\leq 120. \end{aligned}$$

Considering department **II** in a similar way, we get:

Since:	the production of 1 type **A** chair uses 2 hours,
then:	the production of x type **A** chairs takes $2x$ hours.
Similarly:	the production of 1 type **B** chair uses 4 hours,
so:	the production of y type **B** chairs takes $4y$ hours.

The total time used is therefore

$$(2x + 4y) \text{ hours.}$$

Since only 150 hours are available in department $\mathbf{II}$, a second constraint is

$$\begin{aligned} (2x + 4y) \text{ hours} &\leq 150 \text{ hours}, \qquad \text{or just} \\ (2x + 4y) &\leq 150. \end{aligned}$$

We're still not done! We can't make a negative number of chairs, so we also have:

$$\begin{aligned} x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

These are called the **non–negativity constraints**.

Step 3: Objective function

Our objective here is to **maximise income**.

If we make x type **A** chairs, then we get $£10 \times x = £10x$, since each type **A** chair sells for £10.

Similarly, if we make y type **B** chairs, then we get $£12 \times y = £12y$, since each type **B** chair sells for £12.

The total income is then

$$\pounds Z = \pounds(10x + 12y).$$

The aim is to maximise income, so we'd like to maximise

$$Z = 10x + 12y,$$

where Z is the objective function.

Thus, to summarise, we have the following linear programming problem:

Maximise $Z = 10x + 12y$ subject to the constraints

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

Example 2: Replica sports shirts

Sportizus Clothing Company produce replica football shirts and replica rugby shirts for sale on the high street. Each shirt produced goes through a **sewing process** and a **transfer process**.

Each football shirt requires **8** minutes of **sewing time** and **9** minutes for the **transfer process**, whereas rugby shirts each require **5** minutes of **sewing time** and **15** minutes for the **transfer process**.

In any given day, the **total time** available for the sewing process and transfer process is **10** hours and **15** hours respectively.

Example 2: Replica sports shirts

To meet current demand, *Sportizus* must produce at least **30** football shirts and **10** rugby shirts each day.

The company sells football shirts and rugby shirts at a **profit** of £22 and £16 respectively.

How many of each type of shirt should Sportizus produce in order to maximise profits?

Example 2: Replica sports shirts

Let's start off with a table which summarises the question:

	Sewing (mins)	Transfer (minutes)	Profit (P)
Football	**8**	**9**	**22**
Rugby	**5**	**15**	**16**
Total time	**600**	**900**	

Step 1: Decision variables

The decision variables are the number of football and rugby shirts to make. Let

$$x = \text{number in football shirts to make} \qquad \text{and}$$

$$y = \text{number of rugby shirts to make.}$$

Step 2: Constraints

The constraints are:

$$\textbf{sewing}: 8x + 5y \leq 600 \quad \text{and}$$

$$\textbf{transfer}: 9x + 15y \leq 900$$

We also have the **non–negativity contrainits**. However, we are also told that we must make at least 30 football shirts and 10 rugby shirts, giving

$$x \geq 30 \quad \text{and}$$

$$y \geq 10$$

Step 3: Objective function

The aim is to **maximise profit** – call this P.

We know that we make £22 for each football shirt that we make and sell. Since we make (and sell) x football shirts, this will give

$$£22x \qquad \text{profit.}$$

We also know that we make £16 for each rugby shirt we make and sell. Since we make (and sell) y rugby shirts, this will give

$$£16y \qquad \text{profit.}$$

Thus total profit is

$$£P = 22x + 16y,$$

Summary

Thus, to summarise:

Maximise $P = 22x + 16y$ subject to the constraints

$$\begin{aligned} 8x + 5y &\leq 600, \\ 9x + 15y &\leq 900, \\ x &\geq 30 \qquad \text{and} \\ y &\geq 10. \end{aligned}$$

A haulage company

KJB Haulage receives an order to transport **1600** packages. They have **large** vans, which can take **200** packages each, and **small** vans, which can take **80** packages each.

The cost of running each large van on the required journey is £**40** and the cost of running each small van on the same journey is £**20**.

There is a limited budget for the job which requires that not more than £**340** be spent.

It is additionally required that the number of small vans used must not exceed the number of large vans used.

How many of each type of van should be used if costs are to be kept to a minimum?

A haulage company

To summarise:

	Capacity	Cost (in £)
Large van	200	40
Small van	80	20
Limits	1600	340

Step 1: Decision variables

We need to know how many large vans to use and how many small vans to use. Thus, let

$$\ell = \text{number of large vans used} \qquad \text{and}$$

$$s = \text{number of small vans used}$$

Step 2: Constraints

If all the packages are to be transported, then we want the chosen number of vans to be able to transport *at least* 1600 packages in total.

Thus

$$200\ell + 80s \geq 1600.$$

In fact, this can be simplified to

$$5\ell + 2s \geq 40.$$

Step 2: Constraints

What about cost? We need to make sure the total cost does not exceed £340. Thus, reading down the cost column in the summary table gives

$$40\ell + 20s \leq 340, \qquad \text{i.e.}$$

$$2\ell + s \leq 17.$$

Step 2: Constraints

We are also told that the number of small vans used must not exceed the number of large vans used – giving

$$s \leq \ell.$$

Since we can't use a negative number of vans, we also have the **non–negativity constraints**:

$$\begin{aligned} x &\geq 0 \quad \text{and} \\ y &\geq 0. \end{aligned}$$

Step 3: Objective function

The objective is to keep cost to a minimum. Total cost, say £C, is given by $40\ell + 20s$, i.e.

$$\pounds C = 40\ell + 20s.$$

Summary

Thus, to summarise, we want to minimise $C = 40\ell + 20s$ subject to the constraints

$$\begin{aligned} 5\ell + 2s &\geq 40, \\ 2\ell + s &\leq 17, \\ s &\leq \ell, \\ \ell &\geq 0 \qquad \text{and} \\ s &\geq 0. \end{aligned}$$