Lecture 4

HYPOTHESIS TESTS FOR TWO MEANS

Announcement

CBA4 has now gone “live” in **assessed mode**.

After today’s lecture, you should be able to attempt all questions in the CBA.

Deadline: This Friday, 24th February – but really, the “ultimate deadline” is 23:59:59 on **Sunday 26th February**.

Testing two means

If we have **two** independent random samples from **two** populations, we can compare the two sample means (*c.f.* comparing one sample mean to a *proposed value* in the one–sample case).

We use the same framework for hypothesis testing as for the one–sample test; however, the calculations required for the test statistic are slightly different. There are two situations to consider here:

- Are **both** population variances (σ_1^2 **and** σ_2^2) **known**?
- Are **both** population variances **unknown**?

Both population variances (σ_1^2 and σ_2^2) known

1. **State the null hypothesis**

   This time, the null hypothesis is

   $$H_0 \quad : \quad \mu_1 = \mu_2,$$

   i.e. the two population means are equal.

2. **State the alternative hypothesis**

   We usually test against the (**two–tailed**) alternative:

   $$H_1 \quad : \quad \mu_1 \neq \mu_2,$$

   i.e. the population means *are not* equal. However, we might use the **one–tailed** alternatives:

   $$\begin{aligned} H_1 \quad &: \quad \mu_1 > \mu_2, \qquad \text{or} \\ H_1 \quad &: \quad \mu_1 < \mu_2. \end{aligned}$$

Both population variances (σ_1^2 and σ_2^2) known

3. **Calculate the test statistic**
   The test statistic for a two–sample test (when both population variances are known) is

$$z = \frac{|\bar{x}_1 - \bar{x}_2|}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Both population variances (σ_1^2 and σ_2^2) known

4. **Find the p–value**

   This is found from statistical tables; since, in this case, both population variances σ_1^2 and σ_2^2 are *known*, we refer to standard normal tables. As before, we find a range for our *p*–value by comparing our test statistic to the 10%, 5% and 1% critical values.

5. **Form a conclusion**

   Exactly the same again! Use table 2.1 to help you decide what to do! Word your conclusions in the context of the original question.

Both population variances (σ_1^2 and σ_2^2) unknown

In the more likely situation where the population variances are unknown, the test statistic becomes

$$t = \frac{|\bar{x}_1 - \bar{x}_2|}{s \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}},$$

where s is a "pooled standard deviation", and is found as

$$s = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}.$$

Like before, we have to use t–tables to obtain our critical value; the degrees of freedom is now found as $\nu = n_1 + n_2 - 2$.

Example 1

Before a training session for call centre employees a sample of 50 calls to the call centre had an average duration of 5 minutes, whereas after the training session a sample of 45 calls had an average duration of 4.5 minutes.

The population variance is **known** to have been 1.5 minutes before the course and 2 minutes afterwards.

Has the course been effective?

Solution to Example 1

Steps 1 & 2 *(hypotheses)*

$$H_0 \quad : \quad \mu_B = \mu_A \qquad \text{versus}$$

$$H_1 \quad : \quad \mu_B \neq \mu_A$$

Solution to Example 1

Step 3 *(Test statistic)*

We use

$$\begin{aligned} z &= \frac{|\bar{x}_B - \bar{x}_A|}{\sqrt{\frac{\sigma_B^2}{n_B} + \frac{\sigma_A^2}{n_A}}} \\ &= \frac{|5 - 4.5|}{\sqrt{\frac{1.5}{50} + \frac{2}{45}}} \\ &= \frac{0.5}{\sqrt{0.03 + 0.0444}} \\ &= 1.833 \end{aligned}$$

Solution to Example 1

Step 4 *(p–value)*

Since this is case 1 (both population variances known), we use Table 3.1.

Recall that we have a **two–tailed** alternative, and so we get

Significance level	10%	5%	1%
Critical value	1.645	1.96	2.576

Our test statistic $z = 1.833$ shows that our p–value is between 5% and 10%.

Solution to Example 1

Step 5 *(Conclusion)*

From Table 2.1,

- We have **slight** evidence against H_0
- This is not enough to reject and so we **retain** H_0
- There is insufficient evidence to suggest the training course has been effective!

Example 2

A company is interested in knowing if two branches have the same level of average transactions.

The company sample a small number of transactions and calculates the following statistics:

Shop 1	$\bar{x}_1 = 130$	$s_1^2 = 700$	$n_1 = 12$
Shop 2	$\bar{x}_2 = 120$	$s_2^2 = 800$	$n_2 = 15$

Test whether or not the two branches have (on average) the same level of transactions.

Solution to Example 2

Steps 1 and 2 *(hypotheses)*

Our null and alternative hypotheses are:

$$\begin{aligned} H_0 &: \mu_1 = \mu_2 \qquad \text{versus} \\ H_1 &: \mu_1 \neq \mu_2. \end{aligned}$$

Step 3 *(Test statistic)*

Since both population variances are unknown (only the *sample* values are given), the test statistic is

$$t = \frac{|\bar{x}_1 - \bar{x}_2|}{s \times \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

Solution to Example 2

First the **pooled standard deviation**:

$$
\begin{aligned}
s &= \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}} \\
&= \sqrt{\frac{(12-1)\times 700+(15-1)\times 800}{12+15-2}} \\
&= \sqrt{\frac{11\times 700+14\times 800}{25}} \\
&= \sqrt{\frac{7700+11200}{25}} \\
&= \sqrt{\frac{18900}{25}} \\
&= \sqrt{756} = 27.495.
\end{aligned}
$$

Solution to Example 2

Now we put this int our formula for t:

$$\begin{aligned} t &= \frac{|130-120|}{27.495 \times \sqrt{\frac{1}{12}+\frac{1}{15}}} \\ &= \frac{10}{27.495 \times \sqrt{0.15}} \\ &= \frac{10}{10.649} \\ &= 0.939. \end{aligned}$$

Solution to Example 2

Step 4 *(p–value)*

Since both population variances are unknown, we use t–tables to obtain our critical value.

The degrees of freedom, $\nu = n_1 + n_2 - 2$, i.e.
$\nu = 12 + 15 - 2 = 25$.

Under a two–tailed test, and using table 2.3, we get the following critical values:

Significance level	10%	5%	1%
Critical value	1.708	2.060	2.787

Our test statistic $t = 0.939$ lies to the left of the first critical value, and so our p–value is bigger than 10%.

Solution to Example 2

Step 5 *(Conclusion)*

Using Table 2.1,

- We have **no** evidence against H_0
- Therefore, we **retain** H_0
- There is no significant difference between the average level of transactions at the two shops.