Chapter 4: Model solutions

1. This question is similar to the traffic accidents example in the lecture notes. Since we are interested in finding out whether or not the number of customers is the same every day, the null hypothesis is

$$H_0 \quad : \quad \text{The number of customers is the same every day of the week.}$$

Or you might say

$$H_0 \quad : \quad \text{The data are uniform across days.}$$

The alternative hypothesis is

$$H_1 \quad : \quad \text{The number of customers is } \textit{not} \text{ the same every day.}$$

We should now calculate the test statistic. Recall that for goodness–of–fit tests, this is

$$X^2 = \sum \frac{(O-E)^2}{E},$$

where O and E are the observed and expected frequencies (respectively). We already have the observed frequencies; if we assume the null hypothesis is correct, then the expected frequencies for each day are just 100, since the total number of customers observed is 700 (and there are 7 days in the week!). Thus, we have

Day	O	E	$\frac{(O-E)^2}{E}$
Monday	95	100	0.25
Tuesday	81	100	3.61
Wednesday	84	100	2.56
Thursday	120	100	4
Friday	117	100	2.89
Saturday	110	100	1
Sunday	93	100	0.49
Total	700	700	$X^2 = 14.8$

Thus, the test statistic is $X^2 = \sum \frac{(O-E)^2}{E} = 14.8$.

To find the p–value, we use χ^2 tables (table 3.1). The degrees of freedom is

$$\begin{aligned} \nu &= \text{(number of categories after pooling)} - \text{(number of parameters estimated)} - 1 \\ &= 7 - 0 - 1 \\ &= 6. \end{aligned}$$

From table 3.1, we find the following critical values (using 6 degrees of freedom):

Significance level	10%	5%	1%
Critical value	10.64	12.59	16.81

Our test statistic $X^2 = 14.8$ lies in between the critical values 12.59 and 16.81. Thus, the corresponding p–value lies between 1% and 5%.

We conclude that there is moderate evidence against H_0, and so we reject the null hypothesis and go with the alternative; it appears that the number of customers is *not* the same through the week.

2. Since we are looking at the number of complaints in a certain time interval (day), and there is no fixed upper limit to the number of complaints which *could* be made, an appropriate distribution for these data might be the Poisson distribution. If the binomial distribution could be used, we would need a fixed number of trials, each trial having two possible outcomes ("success" and "failure") – we do not have this set–up.

   To see if the Poisson distribution is consistent with our data, we will test the null hypothesis

   $$H_0 \quad : \quad \text{The number of complaints follows a Poisson distribution}$$

   against the alternative

   $$H_0 \quad : \quad \text{The number of complaints does } \textit{not} \text{ follow a Poisson distribution.}$$

   To calculate the test statistic, we need some expected frequencies based on the Poisson distribution. We use the formula

   $$\Pr(X = r) = \frac{\lambda^r \times e^{-\lambda}}{r!}$$

   to find the expected *probabilities*, and then multiply these by the total sample size (494) to obtain the corresponding expected frequencies. To use this formula, we need an estimate of λ. Now recall that, for the Poisson distribution, λ is equal to the mean. Thus, we have

   $$\begin{aligned}\lambda &= \frac{(0 \times 270) + (1 \times 140) + (2 \times 65) + (3 \times 14) + (4 \times 5)}{494} \\ &= 0.672.\end{aligned}$$

   Now we have λ, we can calculate the expected probabilities, i.e.

   $$\begin{aligned}\Pr(X = 0) &= \frac{0.672^0 \times e^{-0.672}}{0!} \\ &= \frac{0.5107}{1} \\ &= 0.5107.\end{aligned}$$

Similarly,

$$\begin{aligned}\Pr(X=1) &= \frac{0.672^1 \times e^{-0.672}}{1!} \\ &= 0.3432,\end{aligned}$$

$$\begin{aligned}\Pr(X=2) &= \frac{0.672^2 \times e^{-0.672}}{2!} \\ &= 0.1153,\end{aligned}$$

$$\begin{aligned}\Pr(X=3) &= \frac{0.672^3 \times e^{-0.672}}{3!} \\ &= 0.0258.\end{aligned}$$

For the 4+ category, we just add up the probabilities we've got so far and subtract from 1, since the entire probability distribution should sum to 1! Thus,

$$\begin{aligned}\Pr(X \geq 4) &= 1-(0.5107+0.3432+0.1153+0.0258) \\ &= 1-0.995 \\ &= 0.005.\end{aligned}$$

Thus, the expected frequencies can be found by multiplying the expected probabilities by the total sample size (494):

Number of complaints	Expected probability	E
0	0.5107	252.2858
1	0.3432	169.5408
2	0.1153	56.9582
3	0.0258	12.7452
4+	0.005	2.47
		494

Notice that the last category (4+) has an expected frequency of 2.47, which is less than 5. However, we can overcome this by pooling the "4+" category with the "3" category (we also need to do this with the observed frequencies). We can now calculate our test statistic:

Number of complaints	O	E	$\frac{(O-E)^2}{E}$
0	270	252.2858	1.2438
1	140	169.5408	5.1472
2	65	56.9582	1.1354
3+	19	15.2152	0.9415
			8.4679

So our test statistic is $X^2 = \sum \frac{(O-E)^2}{E} = 8.4679$.

A range for our p–value can be found from χ^2 tables (table 3.1). Recall that the degrees of freedom, ν, is found as

$$\nu = \text{(number of categories after pooling)} - \text{(number of parameters estimated)} - 1.$$

We have 4 categories after pooling, and we have estimated 1 parameter from the data (λ). Thus,

$$\begin{aligned} \nu &= 4 - 1 - 1 \\ &= 2. \end{aligned}$$

Thus, from table 3.1, we obtain the following critical values:

Significance level	10%	5%	1%
Critical value	4.61	5.99	9.21

Our test statistic $X^2 = 8.4679$ lies in between the two critical values 5.99 and 9.21. Thus, our p–value lies in between 1% and 5%. Thus, there is moderate evidence against H_0, and so we should reject it and go with the alternative. It appears that the number of complaints does *not* follow a Poisson distribution.