Chapter 9

Linear Programming (II)

9.1 Graphical solutions for two–variable problems

In last week's lecture we discussed how to *formulate* a linear programming problem; this week, we consider how to *solve* such problems. Linear programming problems that involve only two decision variables, x and y, may be solved graphically, that is by drawing lines associated with the constraints, identifying a region of possible solutions and then locating a point in this region that has a particular property (i.e. a point which maximises profit or minimises cost). To solve linear programming problems using graphical methods you need to be able to use the following basic mathematical ideas.

9.1.1 Example (A chair manufacturer)

Recall the problem posed in example 8.2.1 (last week):

A manufacturer makes two kinds of chairs, A and B, each of which has to be processed in two departments, I and II. Chair A has to be processed in department I for 3 hours and in department II for 2 hours. Chair B has to be processed in department I for 3 hours and in department II for 4 hours.

The time available in department I in any given month is 120 hours, and the time available in department II, in the same month, is 150 hours.

Chair A has a selling price of £10 and chair B has a selling price of £12.

The manufacturer wishes to maximise his income. How many of each chair should be made in order to achieve this objective? You may assume that all chairs made can be sold.

Last week we considered how to *formulate* this situation as a linear programming problem; using the graphical techniques outlined in section 10.1, we will now consider how to *solve* this problem.

Recall that things are made much easier if we summarise the information given in a table; we can then read down the columns to obtain the inequalities for the linear programming problem. The table below summarises the information given in this particular problem.

Type of chair	Time in dept. I (hours)	Time in dept. II (hours)	Selling price (£)
A	3	2	10
B	3	4	12
Total time available	120	150	

Recall the three steps involved in formulating a linear programming problem:

1. Identify the **decision variables**

2. Identify the **constraints**

3. State the **objective function**

In this example, the decision variables were identified as

$$\begin{aligned} x &= \text{number of type A chairs made} \qquad \text{and} \\ y &= \text{number of type B chairs made.} \end{aligned}$$

The constraints were

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

We then identified that the objective was to maximise the income, which we called Z, where

$$Z = 10x + 12y.$$

So, in summary, the linear programming problem is:

Maximise $Z = 10x + 12y$ subject to the following constraints:

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

To find the *feasible region* for this problem (i.e. the region on a graph which satisfies all of our inequalities), we proceed by indicating, on a diagram, the region for which all of the inequalities hold.

The first inequality is $3x + 3y \leq 120$. To show this on a graph, we first need to plot the line $3x + 3y = 120$.

- When $x = 0$, we have

$$\begin{aligned} 3 \times 0 + 3y &= 120 \quad \text{i.e.} \\ 3y &= 120 \quad \text{i.e.} \\ y &= 40. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 3x + 3 \times 0 &= 120 \quad \text{i.e.} \\ 3x &= 120 \quad \text{i.e.} \\ x &= 40. \end{aligned}$$

These points are plotted on figure 9.1, and the line with equation $3x + 3y = 120$ drawn. Since we want $3x + 3y \leq 120$, our region of interest lies on or below the line, and so we shade out the space above the line.

Now consider the second inequality $2x + 4y \leq 150$. Again, to show this on a diagram, we first need to plot the line $2x + 4y = 150$.

- When $x = 0$, we have

$$\begin{aligned} 2 \times 0 + 4y &= 150 \quad \text{i.e.} \\ 4y &= 150 \quad \text{i.e.} \\ y &= 37.5. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 2x + 4 \times 0 &= 150 \quad \text{i.e.} \\ 2x &= 150 \quad \text{i.e.} \\ x &= 75. \end{aligned}$$

Again, these points are plotted on figure 9.1, and the line with equation $2x + 4y = 150$ drawn. Since we want $2x + 4y \leq 150$, our region of interest lies on or below the line, and so we shade out the space above the line.

On figure 9.1 we also shade out the inadmissible regions for the two non–negativity constraints.

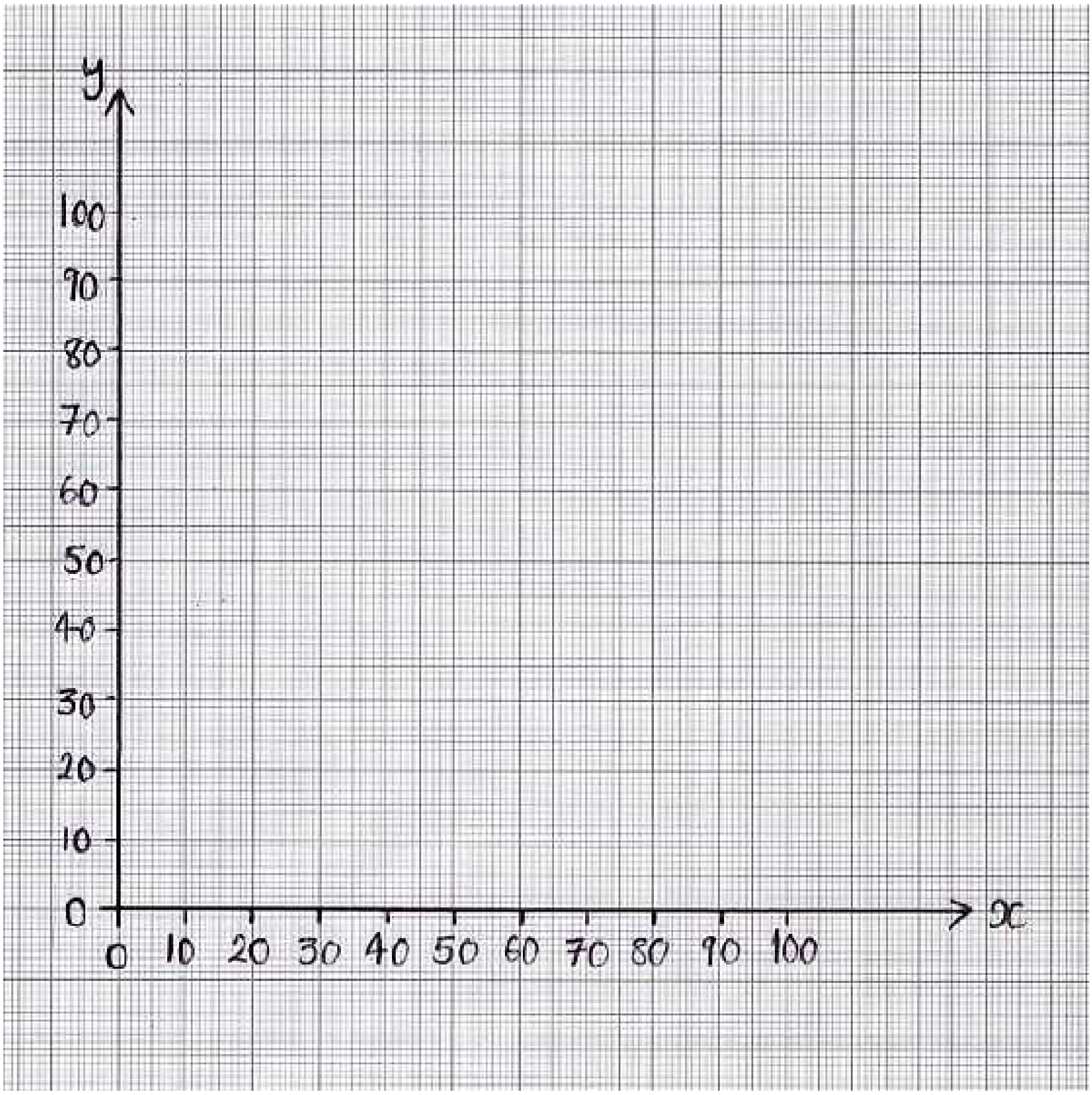

Figure 9.1: Feasible region and objective function for the chair manufacturing problem

The unshaded region in figure 9.1 shows the feasible region associated with our set of inequalities. What we must do now is find the point in that region which meets our objective – i.e. the point in that region which maximises income. One way of doing this is to also plot the objective function. Now our objective function is

$$Z = 10x + 12y,$$

where Z is our income. When Z takes different values we get a family of parallel straight lines. We need to choose a starting value for Z in order to be able to plot the objective function. It's often a good idea to try a value which is a multiple of both the coefficients of x and y. The coefficient of x is 10 and the coefficient of y is 12, so we could try a starting value of $Z = 120$. Thus, the objective function is now

$$10x + 12y = 120.$$

We can plot this line in the same way as before – i.e. consider what happens when x and y are zero.

- When $x = 0$, we have

$$\begin{aligned} 10 \times 0 + 12y &= 120 \quad \text{i.e.} \\ 12y &= 120 \quad \text{i.e.} \\ y &= 10. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 10x + 12 \times 0 &= 120 \quad \text{i.e.} \\ 10x &= 120 \quad \text{i.e.} \\ x &= 12. \end{aligned}$$

This line is also plotted on figure 9.1. Notice that this line does not give the optimal income; the origin represents zero income, and we want to move as far away from this as possible. The largest value of Z (income) will occur at the point in the feasible region that is *furthest* from the origin, but still parallel to the objective line. You can find this point by sliding a ruler over the feasible region so that it is always parallel to the objective function drawn. This will enable you to identify the point that is furthest from the origin. In fact, you should notice that the point in the feasible region furthest from the origin (and parallel to the objective function) is the intersection of the two lines with equations $3x + 3y = 120$ and $2x + 4y = 150$.

All points in the feasible region satisfy our inequalities, but only one point maximises income. Once this point has been identified, we can simply "read off" the x and y values. Doing so give $\boldsymbol{x = 5}$ and $\boldsymbol{y = 35}$, and so, in order to maximise income, we should make **5 type A chairs** and **35 type B chairs**. This will give an income of

$$\begin{aligned} Z &= 10x + 12y \quad \text{i.e.} \\ Z &= 10 \times 5 + 12 \times 35 \quad \text{i.e.} \\ Z &= 50 + 420 \\ &= 470, \end{aligned}$$

i.e. £470.

9.1.2 Example (Replica sports shirts)

Recall the example about *Sportizus*, a company who produce replica football and rugby shirts. Each shirt produced goes through a sewing process and a transfer process.

Each football shirt requires 8 minutes of sewing time and 9 minutes for the transfer process, whereas rugby shirts each require 5 minutes of sewing time and 15 minutes for the transfer process. In any given day, the total time available for the sewing process and transfer process is 10 hours and 15 hours respectively.

To meet current demand, Sportizus must produce at least 30 football shirts and 10 rugby shirts each day. The company sells football shirts and rugby shirts at a profit of £22 and £16 respectively.

How many of each type of shirt should *Sportizus* produce in order to maximise profits?

Last week, we summarised this problem as follows:

Maximise $P = 22x + 16y$ subject to the following constraints:

$$\begin{aligned} 8x + 5y &\leq 600, \\ 9x + 15y &\leq 900, \\ x &\geq 30 \qquad \text{and} \\ y &\geq 10, \end{aligned}$$

where x and y are the number of football and rugby shirts to make, respectively.

The first inequality is $8x + 5y \leq 600$. First, on figure 9.2, let's plot the line $8x + 5y = 600$.

- When $x = 0$, we have

$$\begin{aligned} 8 \times 0 + 5y &= 600 \qquad \text{i.e.} \\ 5y &= 600 \qquad \text{i.e.} \\ y &= 120. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 8x + 5 \times 0 &= 600 \qquad \text{i.e.} \\ 8x &= 600 \qquad \text{i.e.} \\ x &= 75. \end{aligned}$$

So, on figure 9.2, plot the points $(0, 120)$ and $(75, 0)$, and join them up – and then shade out the region *above* the line, since we require $8x + 5y \leq 600$.

The second inequality is $9x + 15y \leq 900$. First, let's plot the line $9x + 15y = 900$.

- When $x = 0$, we have

$$\begin{aligned} 9 \times 0 + 15y &= 900 \quad \text{i.e.} \\ 15y &= 900 \quad \text{i.e.} \\ y &= 60. \end{aligned}$$

- When $y = 0$, we have

$$\begin{aligned} 9x + 15 \times 0 &= 900 \quad \text{i.e.} \\ 9x &= 900 \quad \text{i.e.} \\ x &= 100. \end{aligned}$$

So, on figure 9.2, we now plot the points $(0, 60)$ and $(100, 0)$ and join them up – and then we shade out the area *above* the line, since we require $9x + 15y \leq 900$.

We also require $x \geq 30$ and $y \geq 10$; thus, on figure 9.2, you should also shade out all x values less than 30 and all y values less than 10, to leave $x \geq 30$ and $y \geq 10$ *unshaded*.

Finally, we need to plot the objective function. Here, we have

$$P = 22x + 16y;$$

as with the previous example, we need a starting value for P in order to plot this line. You should notice that starting with $P = 22 \times 16 = 352$ will give $(0, 22)$ and $(16, 0)$, and so we also plot this line on figure 9.2. Notice that this objective line lies outside of the feasible region for this problem. Let's see what happens when we try a bigger starting value for the profit P – for example, let's try a starting value for P four times as big – i.e. $P = 4 \times 352 = 1408$:

- When $x = 0$,

$$\begin{aligned} 22 \times 0 + 16y &= 1408 \quad \text{i.e.} \\ 16y &= 1408 \quad \text{i.e.} \\ y &= 88. \end{aligned}$$

- When $y = 0$,

$$\begin{aligned} 22x + 16 \times 0 &= 1408 \quad \text{i.e.} \\ 22x &= 1408 \quad \text{i.e.} \\ x &= 64. \end{aligned}$$

So now let's try $(0, 88)$ and $(64, 0)$ on figure 9.2. You should notice that part of our line now lies in the feasible region, and that this line lies parallel to our first attempt using $P = 352$. In fact, no matter what value of P we use, we will get a line which lies parallel to our two attempts for the objective line now shown on figure 9.2.

As with the previous example, placing a ruler parallel to the first attempt and sliding it away from the origin (since we want to *maximise* profit), you should see that the last point of contact the ruler makes with the feasible region is, once again, th intersection of our two lines for the main constraints in the problem. Thus, to maximise profit, $\boldsymbol{x} = \mathbf{60}$ and $\boldsymbol{y} = \mathbf{22}$, i.e. we should make **60 football shirts** and **22 rugby shirts** each day. This will give a maximum daily profit of

$$\begin{aligned} P &= 22x + 16y \\ &= 22 \times 60 + 16 \times 22 \\ &= £1672. \end{aligned}$$

Notice also that *minimum profit* will be made when we move our ruler in the opposite direction, i.e. *towards* the origin: this gives $x = 30$ and $y = 10$, and a minimum profit of £820.

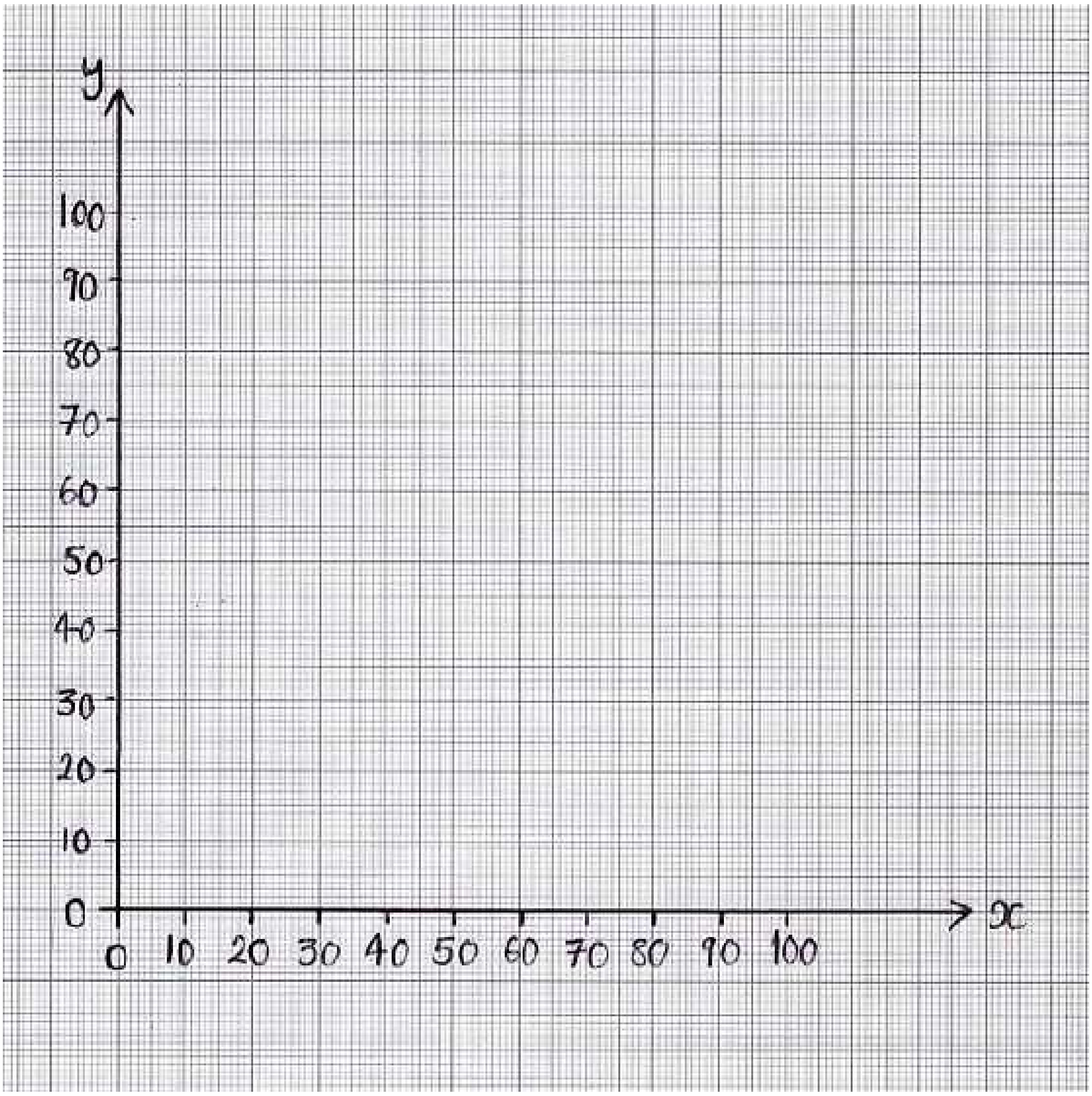

Figure 9.2: Feasible region and objective function for *Sportizus*

9.2 Algebraic solutions for two–variable problems

Notice that in both the examples we have looked at so far, the graphical representation indicates that the solution to these problems is at the intersection of two lines on the graph. Thus, we could find also the solution *algebraically*, as the solution to a pair of *simultaneous equations*.

9.2.1 The chair manufacturer (revisited)

Notice that the solution to the chair manufacturer's problem lay at the intersection of the two lines

$$3x + 3y = 120 \qquad \text{and} \tag{1}$$
$$2x + 4y = 150. \tag{2}$$

We can solve this pair of simultaneous equations by *elimination*. Notice that if we multiply equation (1) by 2 and multiply equation (2) by 3, we get:

$$6x + 6y = 240 \qquad \text{and} \tag{3}$$
$$6x + 12y = 450. \tag{4}$$

Now the x's have the same *coefficient*. Subtracting equation (3) from equation (4) gives

$$\begin{aligned} 0x + 6y &= 210 \qquad \text{and so} \\ 6y &= 210 \qquad \text{i.e.} \\ y &= 35. \end{aligned}$$

Substituting $y = 35$ into equation (1) then gives

$$\begin{aligned} 3x + 3 \times 35 &= 120 \\ 3x + 105 &= 120 \\ 3x &= 120 - 105 \\ 3x &= 15 \\ x &= 5. \end{aligned}$$

Now go back to your graphical solution to this problem in figure 9.1. Notice this is exactly what we got when we solved this problem graphically!

9.2.2 ***Sportizus*** (revisited)

Try this one yourself before the tutorials this week!

Exercises

1. Indicate on a diagram the region for which

$$\begin{aligned} 4x + 3y &\leq 12, \\ 2x + 5y &\leq 10, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

2. Solve the linear programming problem formulated in question 1 of exercises 9. [*Hint: Try a starting value for the profit of* £600]

3. Kuddly Pals Co. Ltd make two types of giant soft toy: bears and cats. The quantity of material needed and the time taken to make each type of toy is given in the table below.

Toy	Material (m^2)	Time (minutes)
Bear	5	12
Cat	8	8

Each day the company can process up to 2000m^2 of material and there are 48 worker hours available to assemble the toys.

The profit made on each bear is £1.50 and on each cat is £1.75. Kuddly Pals Co. Ltd wishes to maximise its daily profit.

(i) Formulate the company's situation as a linear programming problem.

(ii) Draw a suitable diagram to enable the problem to be solved graphically, indicating the feasible region and the direction of the objective line.

(iii) Use your diagram to find the company's maximum profit, £P.

(iv) Verify your solution algebraically.