Chapter 8

Linear programming (I)

8.1 Introduction

Decision making is a process that has to be carried out in many areas of life. Usually there is a particular aim in making one decision rather than another. Two particular aims that are often considered in commerce are maximising profit and minimising loss.

During, and after, the Second World War a group of American mathematicians developed some mathematical methods to help with decision making. They sought to produce mathematical models of situations in which all the requirements, constraints and objectives were expressed as algebraic equations. They then developed methods for obtaining the *optimal solution* – the maximum or minimum value of a required function.

In this chapter we will study problems for which all the algebraic expressions are linear, that is, of the form

$$(\text{a number})x + (\text{a number})y;$$

For example,

$$\text{Profit} \quad = \quad 4x + 3y$$

is a linear equation in x and y, and

$$16x + 18y \quad \leq \quad 25$$

is a linear *inequation* or *inequality* in x and y. This area of mathematics is called *linear programming.*

Linear programming methods are some of the most widely used method employed to solve management and economic problems. They have been applied in a variety of contexts, some of which will be discussed in this chapter, with enormous savings in money and resources.

8.2 Formulating linear programming problems

The first step in formulating a linear programming problem is to determine which quantities you need to know to solve the problem. These are called the *decision variables*. The second step is to decide what the *constraints* are in the problem. For example, there may be a limit on resources or a maximum or minimum value a decision variable may take, or there could be a relationship between two decision variables.

The third step is to determine the objective to be achieved. This is the quantity to be maximised or minimised, that is, *optimised*. The function of the decision variables that is to be *optimised* is called the *objective function*.

The examples which follow illustrate the varied nature of problems that can be modelled by a linear programming model. We will not, at this stage, attempt to solve these problems but instead concentrate on producing the objective function and the constraints, writing these in terms of the decision variables. As an aid to this it is often useful to summarise all the given information in the form of a table (as illustrated in example 8.2.1).

8.2.1 Example (A chair manufacturer)

A manufacturer makes two kinds of chairs, A and B, each of which has to be processed in two departments, I and II. Chair A has to be processed in department I for 3 hours and in department II for 2 hours. Chair B has to be processed in department I for 3 hours and in department II for 4 hours.

The time available in department I in any given month is 120 hours, and the time available in department II, in the same month, is 150 hours.

Chair A has a selling price of £10 and chair B has a selling price of £12.

The manufacturer wishes to maximise his income. How many of each chair should be made in order to achieve this objective? You may assume that all chairs made can be sold.

In this section, **we will not attempt to solve this problem**; we will simply *formulate* the situation as a linear programming problem. You'll notice that there's a lot of information given in the question – this is typical of a linear programming problem. Sometimes it's easier to summarise the information given in a table:

Type of chair	Time in dept. I (hours)	Time in dept. II (hours)	Selling price (£)
A	3	2	10
B	3	4	12
Total time available	120	150	

To formulate this linear programming problem, we consider the following three steps:

1. What are the *decision variables*? (i.e. which quantities do you need to know in order to solve the problem?)

2. What are the *constraints*?

3. What is the *objective*?

Step 1: Decision variables
To find out what the decision variables are, read through the question and identify the things you'd like to know in order to solve the problem. You can usually do this by going straight to the last sentence of the question. The last sentence here is

"How many of each chair should be made..."

Thus, we'd like to know the number of type A chairs to make, and the number of type B chairs to make. These are our decision variables, and are usually denoted with lower case letters (x and y if we have two decision variables, x, y and z if we have three, for example). Thus, our decision variables are

$$\begin{aligned} x &= \text{number of type A chairs made} \qquad \text{and} \\ y &= \text{number of type B chairs made.} \end{aligned}$$

Step 2: Constraints
Identifying the constraints is probably the hardest bit. To understand this bit, consider what could happen in each department. For example, if we focus on what could happen in department I:

Since:	the production of 1 type A chair uses 3 hours,
then:	the production of x type A chairs takes $3 \times x = 3x$ hours.
Similarly:	the production of 1 type B chair uses 3 hours,
so:	the production of y type B chairs takes $3 \times y = 3y$ hours.

The total time used is therefore

$$(3x + 3y) \text{ hours.}$$

Since only 120 hours are available in department I, one constraint is

$$\begin{aligned} (3x + 3y) \text{ hours} &\leq 120 \text{ hours}, \qquad \text{or just} \\ (3x + 3y) &\leq 120. \end{aligned}$$

Considering department II in a similar way, we get:

Since:	the production of 1 type A chair uses 2 hours,
then:	the production of x type A chairs takes $2 \times x = 2x$ hours.
Similarly:	the production of 1 type B chair uses 4 hours,
so:	the production of y type B chairs takes $4 \times y = 4y$ hours.

The total time used is therefore

$$(2x + 4y) \text{ hours.}$$

Since only 150 hours are available in department II, a second constraint is

$$\begin{aligned} (2x + 4y) \text{ hours} &\leq 150 \text{ hours}, \qquad \text{or just} \\ (2x + 4y) &\leq 150. \end{aligned}$$

In addition to the two constraints we have identified above, we also require that x and y are non–negative (since we can't make a negative number of chairs!), and so we also have the following two constraints:

$$\begin{aligned} x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

These are called the *non–negativity constraints*.

Step 3: Objective function
Our objective here is to maximise income. If we make x type A chairs, then we get $\pounds 10 \times x = \pounds 10x$, since each type A chair sells for £10.

Similarly, if we make y type B chairs, then we get $\pounds 12 \times y = \pounds 12y$, since each type B chair sells for £12.

The total income is then

$$\pounds Z = \pounds(10x + 12y).$$

The aim is to maximise income, so we'd like to maximise

$$Z = 10x + 12y,$$

where z is the objective function.

Thus, to summarise, we have the following linear programming problem:

Maximise $Z = 10x + 12y$ subject to the constraints

$$\begin{aligned} 3x + 3y &\leq 120, \\ 2x + 4y &\leq 150, \\ x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

8.2.2 Example (Replica sports shirts)

Sportizus Clothing Company produce replica football shirts and replica rugby shirts for sale on the high street. Each shirt produced goes through a sewing process and a transfer process.

Each football shirt requires 8 minutes of sewing time and 9 minutes for the transfer process, whereas rugby shirts each require 5 minutes of sewing time and 15 minutes for the transfer process. In any given day, the total time available for the sewing process and transfer process is 10 hours and 15 hours respectively.

To meet current demand, Sportizus must produce at least 30 football shirts and 10 rugby shirts each day. The company sells football shirts and rugby shirts at a profit of £22 and £16 respectively.

How many of each type of shirt should *Sportizus* produce in order to maximise profits?

Let's start off with a table which summarises the question:

	Sewing time (mins)	Transfer process (minutes)	Profit (P)
Football			
Rugby			
Total time			

Notice that the time available has been converted to minutes to be consistent with the other times given.

Step 1: Decision variables

The decision variables are the number of football and rugby shirts to make. Let

x = and

y =

Step 2: Constraints

The constraints are:

sewing : and

transfer :

We do, of course, also have the non–negativity conditions; however, we are also told that we must make at least 30 football shirts and 10 rugby shirts to meet demand, giving:

$x \geq$ and

$y \geq$

Step 3: Objective function

Thus, to summarise, we have:

8.2.3 Example (A haulage company)

KJB Haulage receives an order to transport 1600 packages. They have large vans, which can take 200 packages each, and small vans, which can take 80 packages each.

The cost of running each large van on the required journey is £40 and the cost of running each small van on the same journey is £20.

There is a limited budget for the job which requires that not more than £340 be spent. It is additionally required that the number of small vans used must not exceed the number of large vans used.

How many of each type of van should be used if costs are to be kept to a minimum?

The following table summarises the given information:

	Capacity	Cost (in £)
Large van	200	40
Small van	80	20
Limits	1600	340

Step 1: Decision variables
The decision required is how many of each type of van to use to fulfil the order. The decision variables are thus:

$$\begin{aligned} l &= \text{number of large vans used} \qquad \text{and} \\ s &= \text{number of small vans used.} \end{aligned}$$

Step 2: Constraints
If all the packages are to be transported, then we want the chosen number of large vans and small vans to be able to transport at least 1600 packages in total.

Since one large van can transport 200 packages, l large vans can transport $200 \times l = 200l$ packages. One small van can transport 80 packages; thus, s small vans can transport $80 \times s = 80s$ packages. The total number of packages transported is thus

$$200l + 80s,$$

and since the total number of packages must be at least 1600, our first inequality is

$$200l + 80s \geq 1600.$$

Since 40 goes into each of the numbers in the above inequality, we could simplify by dividing by 40, to give

$$5l + 2s \geq 40.$$

We now turn our attention to cost. Since one large van costs £40, l large vans will cost £$(40 \times l)$ = £$40l$. Similarly, one small van costs £20, so s small vans will cost £$(20 \times s)$ = £$20s$. Thus, the total cost of the order will be

$$40l + 20s.$$

Since the total cost of the order must not exceed £340, our second inequality is

$$40l + 20s \leq 340.$$

Again, this inequality may be simplified – this time by dividing through by 20. Thus, we have

$$2l + s \leq 17.$$

We are also told that the number of small vans must not exceed the number of large vans used. Thus, a third inequality is

$$s \leq l.$$

Since we can't have a negative number of vans (either large or small), we also have the non–negativity constraints

$$\begin{aligned} x &\geq 0 \qquad \text{and} \\ y &\geq 0. \end{aligned}$$

Step 3: Objective function

The objective is to keep costs to a minimum. Recall that the total cost is given by $40l + 20s$. If we let the total cost be £C, the objective function is

$$\pounds C = 40l + 20s.$$

Thus, to summarise, the linear programming problem is therefore:

Minimise $C = 40l + 20s$ subject to the constraints

$$\begin{aligned} 5l + 2s &\geq 40, \\ 2l + s &\leq 17, \\ s &\leq l, \\ l &\geq 0 \qquad \text{and} \\ s &\geq 0. \end{aligned}$$

Notice that we have discussed how to *formulate* linear programming problems, but have yet to *solve* them. There are a variety of techniques available to this end; by far the simplest are graphical solutions, which we will study next week.

Exercises

1. A factory is to install two types of machine – A and B. Type A requires one operator and occupies 3m^2 of floor space. Type B requires 2 operators and occupies 4m^2 of floor space.

   The maximum number of operators available is 40 and the total floor space available is 100m^2.

   Given that the weekly profits on type A and type B machines are £75 and £120 (respectively), formulate a linear programming problem to find the number of each machine that should be bought in order to maximise the profit. *Do not attempt to solve this problem.*

   Hint: to identify the inequalities, try completing the following table:

	Number of operators	Floor space (m^2)
Type A		
Type B		
Limits		

2. A small toy manufacturer makes 2 kinds of toys, A and B. Each type of toy spends time on a machine, and time with a craftsman who adds detail by hand.

   Each of toy A takes up 3 hours of machine time and 2 hours of the craftsman's time. Each of toy B takes up 3 hours of machine time and 4 hours of the craftsman's time.

   The total machine time available each week is 40 hours. The manufacturer employs two part–time craftsmen who both work 25 hours each week – the manufacturer operates a family–friendly policy of no overtime.

   The manufacturer will make £10 profit on each of toy A sold, and £12 profit on each of toy B. Formulate this scenario as a linear programming problem which will determine how many of each type of toy should be made in order to maximise the manufacturer's profit. *Do not attempt to solve this problem.*