Logarithms

Definition: The logarithm of a number to a given base is the power to which that base must be raised to give the number.

In symbols, if $x = b^z$ then $z = \log_b x$.

The fundamental result

Write $b^m = u$, $b^n = v$ and $b^{m+n} = w$. Then since $b^m \times b^n = b^{m+n}$, $u \times v = w$.
[e.g. $4 \times 8 = 2^2 \times 2^3 = 2^5 = 32$]
But by definition $m = \log_b u$, $n = \log_b v$ and $m + n = \log_b w$ so that
$\log_b (u \times v) = \log_b u + \log_b v$.
In words, taking logarithms converts variables which have multiplicative effects into ones which have additive effects.

It follows that for integer values of *k*, $\log_b x^k = k \log_b x$.
This is true whether *k* is an integer or not; e.g. $\log \sqrt{x} \equiv \log x^{1/2} = \frac{1}{2} \log x$.

Two special cases: $\log_b b = 1$; $\log_b 1 = 0$ [$\log\left(u \frac{1}{u}\right) = \log u + (-1) \log u = 0$].

Converting from one base to another

Write $z = \log_b x$ and take logarithms to base *c* of each side of the implied equation $b^z = x$, getting $\log_c b^z = \log_c x$, whence $z \log_c b = \log_c x$. Therefore $\log_c x = \log_c b \times \log_b x$. In other words, the logarithms of a set of numbers to one base are just constant multiples of their logarithms to another base.

The only frequently used bases are 2, 10 and *e* = 2.71828... .The first of these is convenient when doses of substances are doubled in an experiment, the second gives a clearer view of the size of the number whose logarithm has been taken (because if $m < \log_{10} x < m + 1$, then $10^m < x < 10^{m+1}$). The third is known as the base of "natural" logarithm, and $\log_e x$ is sometimes written $\ln x$. Also e^x can be written as $\exp x$.

$\log(\exp x) = \exp(\log x) = x$.

Exponential decay

If the concentration of a drug goes down with time following the equation $y = y_0 e^{-at}$ it has a constant halving time of $\dfrac{\ln 2}{a}$. To show this suppose that $y = c$ when $t = t_1$ and $y = \frac{1}{2} c$ when $t = t_1 + t$. Then $y_0 e^{-a t_1} = 2 y_0 e^{-a(t+t)}$. Since y_0 appears on both sides of this equation, it disappears, and taking logarithms gives $-a t_1 = \ln 2 - a(t_1 + t)$.

Now at_1 is on both sides of the equation and we get $t = \frac{\ln 2}{a}$ as required, showing that the halving time is independent of time or concentration.

If we work with $\ln y$ instead of y the equation becomes very simple: $\ln y = \ln y_0 - at$; the log concentration decreases linearly with time. A variable which changes by a constant *ratio* in a given time has been transformed to one which changes by a constant *amount*.

Notice that logarithms can be negative; clearly the log concentration in the above equation will be after a long enough time. Indeed if $x < 1$ then $\log x < 0$ for any base. If $\log x = z$ then $\log \frac{1}{x} = -z$.

```
  1     0 7
  6     0 89999
 23     1 00000000011111111
 49     1 22222222222222223333333333
 49     1 4444444445555555555555
 27     1 666677777
 18     1 8888899
 11     2 01111
  6     2 22333
  1     2
  1     2
  1     2
  1     3
  1     3
  1     3 5
```

Creatinine

```
   2    18 78
   3    19 3
   9    19 566889
  20    20 00012233344
  39    20 5556666777777899999
 (16)   21 0001122223333444
  43    21 5666777777888888
  27    22 0001344444
  17    22 556678
  11    23 1222334
   4    23 567
   1    24
   1    24
   1    25 4
```

log Creatinine

Transforming positively skew distributions to Normality.

The following two stem-leaf diagrams show the distribution of creatinine in 98 people, untransformed and after taking "common" logs (to the base 10). The transformed data are much nearer Normality.

Rule of thumb

If you have a variable for which different individuals have values which differ by orders of magnitude, it may well be appropriate to take logs.