

## Gradient, Divergence and Curl

### Question 1

(a) Find the directional derivative of the field  $f = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$  at a point  $(x, y, z)$  in the direction of the vector  $\mathbf{r} = (x, y, z)$ . Here  $a$ ,  $b$ , and  $c$  are constants.

(b) In what case is the directional derivative of part (a) equal to the modulus of the gradient?

### Question 2

Find the directional derivative of a two-dimensional scalar field  $f = x^2 + y^2$  at a point  $(1, 1)$  in the directions of the following vectors:

(a)  $\mathbf{b} = (1, 1)$ , (b)  $\mathbf{b} = (0, 1)$ , and (c)  $\mathbf{b} = (-1, 1)$ .

(d) Also find  $\nabla f$  at  $(x, y) = (1, 1)$  and compare the above directional derivatives with  $|\nabla f|$ .

### Question 3

Find  $\nabla \cdot \mathbf{v}$  for the following vector fields:

(a)  $\mathbf{v} = (y^2 + z^2)(x + y)\hat{\mathbf{x}} + (z^2 + x^2)(y + z)\hat{\mathbf{y}} + (x^2 + y^2)(z + x)\hat{\mathbf{z}}$ ,

(b)  $\mathbf{v} = f_1(y, z)\hat{\mathbf{x}} + f_2(x, z)\hat{\mathbf{y}} + f_3(x, y)\hat{\mathbf{z}}$ ,

(c)  $\mathbf{v} = xf_1(y, z)\hat{\mathbf{x}} + yf_2(x, z)\hat{\mathbf{y}} + zf_3(x, y)\hat{\mathbf{z}}$ .

### Question 4

Find (a)  $\nabla \cdot \mathbf{r}$ , (b)  $\nabla \cdot \frac{\mathbf{r}}{r}$ , (c)  $\nabla \cdot (r^4 \mathbf{r})$ , where  $r = |\mathbf{r}|$  and  $\mathbf{r} = (x, y, z)$ .

### Question 5

Find  $\nabla \times \mathbf{v}$  for  $\mathbf{v} = \frac{y}{x}\hat{\mathbf{x}} + \frac{z}{y}\hat{\mathbf{y}} + \frac{x}{z}\hat{\mathbf{z}}$  at the points (a)  $(x, y, z)$  and (b)  $(-1, -1, -1)$ .

### Question 6

Find  $\nabla \times \mathbf{v}$  at a point  $(1, 1, 2)$  for

(a)  $\mathbf{v} = yz^2\hat{\mathbf{y}} + x\hat{\mathbf{z}}$ , (b)  $\mathbf{v} = yz^2\hat{\mathbf{x}} + x\hat{\mathbf{y}}$ , (c)  $\mathbf{v} = yz^2\hat{\mathbf{z}} + x\hat{\mathbf{x}}$ .

### Question 7

Find  $\nabla \times \mathbf{v}$  for the following vector fields:

(a)  $\mathbf{v} = y^2z^3\hat{\mathbf{x}} + 2xyz^2\hat{\mathbf{y}} + 3xy^2z^2\hat{\mathbf{z}}$ , (b)  $\mathbf{v} = yz\hat{\mathbf{x}} + z(x + 2y)\hat{\mathbf{y}} + y(x + y)\hat{\mathbf{z}}$ ,

(c)  $\mathbf{v} = \frac{y}{x^2}\hat{\mathbf{y}} - \frac{1}{x}\hat{\mathbf{z}}$ , (d)  $\mathbf{v} = \frac{y}{x^2}\hat{\mathbf{x}} - \frac{1}{x}\hat{\mathbf{y}}$ ,

(e)  $\mathbf{v} = \frac{y}{x^2}\hat{\mathbf{z}} - \frac{1}{x}\hat{\mathbf{x}}$ .

### Question 8

Find (a)  $\nabla \times \mathbf{r}$ , (b)  $\nabla \times \frac{\mathbf{r}}{r}$ , (c)  $\nabla \times \frac{\mathbf{r}}{r^3}$ , where  $r = |\mathbf{r}|$  and  $\mathbf{r} = (x, y, z)$ .

### Question 9

Find a function  $g(z)$  such that the vector field  $(2y, 2x + g(z), 3yz^2) = 0$  is irrotational.

### Question 10

Which of the following vector fields are irrotational?

(a)  $(2xyz, x^2z, x^2y)$ ,      (b)  $(0, 1, xy)$ .

### Question 11

Which of the fields of Question 10 are solenoidal?

### Question 12

Consider a scalar field  $f(x, y, z)$ , where  $x$ ,  $y$  and  $z$  lie on a curve  $t \mapsto (x(t), y(t), z(t))$ . Show that  $\frac{df}{dt} = (\mathbf{v} \cdot \nabla)f$ , where  $\mathbf{v}$  is tangent to the curve.

### Question 13

Prove that

$$\nabla \times (\mathbf{c} \times f(r)\mathbf{r}) = 2f(r)\mathbf{c} + \frac{1}{r} \frac{df}{dr} [\mathbf{c}r^2 - \mathbf{r}(\mathbf{c} \cdot \mathbf{r})].$$

Here  $f(x, y, z)$  is a scalar field,  $\mathbf{r} = (x, y, z)$  is the position vector,  $r = |\mathbf{r}|$ , and  $\mathbf{c}$  is a constant vector.

## Solutions

Note: The Solutions presented below are given in both coordinate and subscript notation. The latter has only been presented in a handout and is OPTIONAL. You are free to use any method you wish when solving the problems.

### Solution 1

(a) We first calculate the gradient of  $f$  (it is useful to take advantage of the symmetry of  $f$  in  $x$ ,  $y$  and  $z$ , but note that  $x$  combines with  $a$ ,  $y$  with  $b$  and  $z$  with  $c$ ):

$$\nabla f = \frac{2x}{a^2} \hat{\mathbf{x}} + \frac{2y}{b^2} \hat{\mathbf{y}} + \frac{2z}{c^2} \hat{\mathbf{z}}.$$

The unit vector in the direction  $\mathbf{r} = (x, y, z)$  is  $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x}{r} \hat{\mathbf{x}} + \frac{y}{r} \hat{\mathbf{y}} + \frac{z}{r} \hat{\mathbf{z}}$ , where  $r = \sqrt{x^2 + y^2 + z^2}$ . Hence,

$$\frac{\partial f}{\partial r} = \hat{\mathbf{r}} \cdot \nabla f = \frac{2}{r} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = \frac{2f}{r}.$$

Otherwise,  $f = \frac{x_i^2}{a_i^2}$ , where  $a_1 = a$ ,  $a_2 = b$ ,  $a_3 = c$ . So,  $\nabla f = \mathbf{e}_i \frac{2x_i}{a_i^2}$ . With  $\hat{\mathbf{r}} = x_i/r$ , we obtain

$$\nabla f \cdot \hat{\mathbf{r}} = \frac{2x_i}{a_i^2} \frac{x_i}{r} = 2 \frac{x_i^2}{a_i^2} \frac{1}{r} = 2 \frac{f}{r}.$$

(b) The modulus of the gradient is given by  $|\nabla f| = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$  and this is equal to  $\partial f/\partial r$  only if  $a = b = c$ , and then both are equal to  $2r/a^2$ .

To understand this result, we can observe that  $\nabla f$  is a normal to the surface  $f(x, y, z) = \text{const}$ . For  $f$  given above, this surface is an ellipsoid, which reduces to a sphere if only  $a = b = c$ . The directional derivative can be equal to the gradient if only the direction  $\hat{\mathbf{b}}$ , along which the derivative is calculated, is the direction of the normal vector. However,  $\mathbf{r}$  is a normal to an ellipsoidal surface only if the surface is in fact a sphere,  $a = b = c$ .

### Solution 2

We first find the gradient of  $f$ :  $\nabla f = 2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}}$ . At a point  $(1, 1)$ , we have  $\nabla f(1, 1) = 2\hat{\mathbf{x}} + 2\hat{\mathbf{y}}$ .

Using our definition of the directional derivative, we obtain:

$$(a) \quad \frac{\partial f}{\partial b} = (2\hat{\mathbf{x}} + 2\hat{\mathbf{y}}) \cdot \frac{1\hat{\mathbf{x}} + 1\hat{\mathbf{y}}}{\sqrt{1^2 + 1^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2},$$

$$(b) \quad \frac{\partial f}{\partial b} = (2\hat{\mathbf{x}} + 2\hat{\mathbf{y}}) \cdot \frac{0\hat{\mathbf{x}} + 1\hat{\mathbf{y}}}{\sqrt{0^2 + 1^2}} = \frac{2}{\sqrt{1}} = 2,$$

$$(c) \quad \frac{\partial f}{\partial b} = (2\hat{\mathbf{x}} + 2\hat{\mathbf{y}}) \cdot \frac{-1\hat{\mathbf{x}} + 1\hat{\mathbf{y}}}{\sqrt{(-1)^2 + 1^2}} = \frac{0}{\sqrt{2}} = 0.$$

(d)  $|\nabla f(1, 1)| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ . The modulus of the gradient is equal to the directional derivative in (a). This is not surprising since here  $\mathbf{b} = (1, 1)$  is parallel to  $\nabla f(1, 1) = (2, 2)$ .

The modulus of the gradient is larger than the directional derivative in (b) and (c); this must be so since  $|\nabla f|$  is the maximum rate of change of  $f$ .

Note that the directional derivative is zero in (c). It is easy to verify that  $\mathbf{b} = (-1, 1)$  is orthogonal to  $\nabla f = (2, 2)$ : indeed,  $\mathbf{b} \cdot \nabla f = -1 \cdot 2 + 1 \cdot 2 = 0$ . Since  $\nabla f$  is a normal to a surface  $f = \text{const}$ ,  $\mathbf{b}$  must be tangent to the surface. We conclude that the directional derivative is zero in those directions which are tangent to the surface.

**Solution 3**

$$(a) \quad \nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}(y^2 + z^2)(x+y) + \frac{\partial}{\partial y}(z^2 + x^2)(y+z) + \frac{\partial}{\partial z}(x^2 + y^2)(z+x) = (y^2 + z^2) + (z^2 + x^2) + (x^2 + y^2) \\ = 2(x^2 + y^2 + z^2) = 2r^2.$$

$$(b) \quad \nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}f_1(y, z) + \frac{\partial}{\partial y}f_2(x, z) + \frac{\partial}{\partial z}f_3(x, y) = 0 + 0 + 0 = 0.$$

$$(c) \quad \nabla \cdot \mathbf{v} = \frac{\partial}{\partial x}[xf_1(y, z)] + \frac{\partial}{\partial y}[yf_2(x, z)] + \frac{\partial}{\partial z}[zf_3(x, y)] = f_1(y, z) + f_2(x, z) + f_3(x, y).$$

**Solution 4**

Taking advantage of the symmetry of expressions containing  $r$  can significantly accelerate calculations and help to avoid errors.

$$(a) \quad \nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3.$$

Otherwise,  $\nabla \cdot \mathbf{r} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 1 + 1 + 1 = 3$  (remember the summation convention).

$$(b) \quad \nabla \cdot \frac{\mathbf{r}}{r} = \frac{\partial}{\partial x} \frac{x}{r} + \frac{\partial}{\partial y} \frac{y}{r} + \frac{\partial}{\partial z} \frac{z}{r} = \frac{r - (x/r)x}{r^2} + \frac{r - (y/r)y}{r^2} + \frac{r - (z/r)z}{r^2} \\ = \frac{3r - (x^2 + y^2 + z^2)/r}{r^2} = \frac{3r - r}{r^2} = \frac{2}{r} \quad (\text{we recall that, e.g., } \partial r / \partial x = x/r).$$

Alternatively,  $\nabla \cdot \frac{\mathbf{r}}{r} = \frac{\partial}{\partial x_i} \frac{x_i}{r} = \frac{r - x_i x_i / r}{r^2}$  since  $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$ . Thus,

$$\nabla \cdot \frac{\mathbf{r}}{r} = \frac{r - 3r}{r^2} = -\frac{2}{r} \quad \text{since } x_i x_i = r^2.$$

$$(c) \quad \nabla \cdot (r^4 \mathbf{r}) = \frac{\partial}{\partial x}(r^4 x) + \frac{\partial}{\partial y}(r^4 y) + \frac{\partial}{\partial z}(r^4 z) = 4r^3 \frac{x}{r} + r^4 + 4r^3 \frac{y}{r} + r^4 + 4r^3 \frac{z}{r} + r^4 \\ = 4r^2(x^2 + y^2 + z^2) + 3r^4 = 7r^4.$$

Alternatively,  $\nabla \cdot (r^4 \mathbf{r}) = \frac{\partial}{\partial x_i} r^4 x_i = x_i \frac{\partial}{\partial x_i} r^4 + r^4 \frac{\partial}{\partial x_i} x_i = x_i 4r^3 \frac{x_i}{r} + 3r^4 = 4r^4 + 3r^4 = 7r^4$ . Note that  $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$ ,  $x_i x_i = r^2$  and  $\frac{\partial x_i}{\partial x_i} \delta_{ii} = 3$  because of the summation convention.

**Solution 5**

In this example we can take advantage of another, more subtle form of symmetry. The three components of the vector  $\mathbf{v}$  in this question can be obtained from each other by the CYCLIC PERMUTATIONS of  $x$ ,  $y$  and  $z$  according to the following pattern:  $x \rightarrow y$ ,  $y \rightarrow z$ ,  $z \rightarrow x$ . The order here corresponds to the standard ordering of the coordinates  $x$ ,  $y$  and  $z$ .

In other words, if in the  $x$ -component  $(y/x)\hat{\mathbf{x}}$  we replace  $x$  by  $y$  and  $y$  by  $z$ , we obtain the  $y$ -component  $(z/y)\hat{\mathbf{y}}$ , and replacing here  $y$  by  $z$  and  $z$  by  $x$ , we obtain the  $z$ -component of  $\mathbf{v}$ . It can be easily verified (just inspect our basic definitions) that the operator  $\nabla$  and the cross product also possess this symmetry. Therefore, it is sufficient to calculate only ONE component of the curl, and the remaining two will follow via cyclic permutations.

(a) Applying the definition of curl, we have

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y/x & z/y & x/z \end{vmatrix} = \hat{\mathbf{x}} \left( \frac{\partial}{\partial y} \frac{x}{z} - \frac{\partial}{\partial z} \frac{z}{y} \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} \frac{y}{x} - \frac{\partial}{\partial x} \frac{x}{z} \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x} \frac{z}{y} - \frac{\partial}{\partial y} \frac{y}{x} \right) \\ = -\frac{1}{y}\hat{\mathbf{x}} - \frac{1}{z}\hat{\mathbf{y}} - \frac{1}{x}\hat{\mathbf{z}}.$$

(b) For  $x = -1$ ,  $y = -1$  and  $z = -1$ , we obtain  $\nabla \times \mathbf{v} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}} = \mathbf{r}$ .

### Solution 6

(a)

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & yz^2 & x \end{vmatrix} = \hat{\mathbf{x}} \left( \frac{\partial}{\partial y} x - \frac{\partial}{\partial z} yz^2 \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} 0 - \frac{\partial}{\partial x} x \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x} yz^2 - \frac{\partial}{\partial y} 0 \right) = -2yz\hat{\mathbf{x}} - \hat{\mathbf{y}}.$$

Thus,  $\nabla \times \mathbf{v}(1, 1, 2) = -4\hat{\mathbf{x}} - \hat{\mathbf{y}}$ .

(b)

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & x & 0 \end{vmatrix} = \hat{\mathbf{x}} \left( \frac{\partial}{\partial y} 0 - \frac{\partial}{\partial z} x \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} yz^2 - \frac{\partial}{\partial x} 0 \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x} x - \frac{\partial}{\partial y} yz^2 \right) \\ &= 2yz\hat{\mathbf{y}} + (1 - z^2)\hat{\mathbf{z}}. \end{aligned}$$

Thus,  $\nabla \times \mathbf{v}(1, 1, 2) = 4\hat{\mathbf{y}} - 3\hat{\mathbf{z}}$ .

(c)

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & yz^2 \end{vmatrix} = \hat{\mathbf{x}} \left( \frac{\partial}{\partial y} yz^2 - \frac{\partial}{\partial z} 0 \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} x - \frac{\partial}{\partial x} yz^2 \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x} 0 - \frac{\partial}{\partial y} x \right) = z^2\hat{\mathbf{x}}.$$

Thus,  $\nabla \times \mathbf{v}(1, 1, 2) = 4\hat{\mathbf{x}}$ .

### Solution 7

(a) There is no symmetry with respect to cyclic permutations in this example [and also in parts (b)–(e) below], so each component must be evaluated separately.

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^2 & 3xy^2z^2 \end{vmatrix} \\ &= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y} 3xy^2z^2 - \frac{\partial}{\partial z} 2xyz^2 \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} y^2z^3 - \frac{\partial}{\partial x} 3xy^2z^2 \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x} 2xyz^2 - \frac{\partial}{\partial y} y^2z^3 \right) \\ &= (6xyz^2 - 4xyz)\hat{\mathbf{x}} + (3y^2z^2 - 3y^2z^2)\hat{\mathbf{y}} + (2yz^2 - 2yz^3)\hat{\mathbf{z}} \\ &= 2xyz(3z - 2)\hat{\mathbf{x}} + 2yz^2(1 - z)\hat{\mathbf{z}}. \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & z(x+2y) & y(x+y) \end{vmatrix} \\ &= \hat{\mathbf{x}} \left( \frac{\partial}{\partial y} y(x+y) - \frac{\partial}{\partial z} z(x+2y) \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} yz - \frac{\partial}{\partial x} y(x+y) \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x} z(x+2y) - \frac{\partial}{\partial y} yz \right) \\ &= (x+2y-x-2y)\hat{\mathbf{x}} + (y-y)\hat{\mathbf{y}} + (z-z)\hat{\mathbf{z}} = 0. \end{aligned}$$

$$(c) \quad \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & y/x^2 & -1/x \end{vmatrix} = \hat{\mathbf{x}} \left( -\frac{\partial}{\partial y} \frac{1}{x} - \frac{\partial}{\partial z} \frac{y}{x^2} \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} 0 + \frac{\partial}{\partial x} \frac{1}{x} \right) + \hat{\mathbf{z}} \left( \frac{\partial}{\partial x} \frac{y}{x^2} - \frac{\partial}{\partial y} 0 \right)$$

$$= -\frac{1}{x^2} \hat{\mathbf{y}} - \frac{2y}{x^3} \hat{\mathbf{z}}.$$

$$(d) \quad \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y/x^2 & -1/x & 0 \end{vmatrix} = \hat{\mathbf{x}} \left( -\frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} \frac{1}{x} \right) + \hat{\mathbf{y}} \left( \frac{\partial}{\partial z} \frac{y}{x^2} - \frac{\partial}{\partial x} 0 \right) + \hat{\mathbf{z}} \left( -\frac{\partial}{\partial x} \frac{1}{x} - \frac{\partial}{\partial y} \frac{y}{x^2} \right)$$

$$= 0.$$

$$(e) \quad \nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -1/x & 0 & y/x^2 \end{vmatrix} = \hat{\mathbf{x}} \left( \frac{\partial}{\partial y} \frac{y}{x^2} - \frac{\partial}{\partial z} 0 \right) + \hat{\mathbf{y}} \left( -\frac{\partial}{\partial z} \frac{1}{x} - \frac{\partial}{\partial x} \frac{y}{x^2} \right) + \hat{\mathbf{z}} \left( -\frac{\partial}{\partial x} 0 + \frac{\partial}{\partial y} \frac{1}{x} \right)$$

$$= \frac{1}{x^2} \hat{\mathbf{x}} + \frac{2y}{x^3} \hat{\mathbf{y}}.$$

### Solution 8

Symmetry with respect to cyclic permutations can be employed here with a very rewarding effect.

$$(a) \quad \nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\mathbf{x}} \left( \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) + \hat{\mathbf{y}} \left( \frac{\partial x}{\partial z} - \frac{\partial z}{\partial x} \right) + \hat{\mathbf{z}} \left( \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = 0.$$

Otherwise, using subscript notation,  $\nabla \times \mathbf{r} = \varepsilon_{ijk} \frac{\partial x_k}{\partial x_j} = \varepsilon_{ijk} \delta_{jk} = \varepsilon_{ijj} = 0$ .

$$(b) \quad \text{First note that, say, } \frac{\partial}{\partial z} \frac{y}{r} = y \frac{\partial}{\partial z} \frac{1}{r} = -y \frac{1}{r^2} \frac{\partial r}{\partial z} = -y \frac{1}{r^2} \frac{z}{r} = -\frac{yz}{r^3}.$$

The  $x$ -component of  $\nabla \times (\mathbf{r}/r) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x/r & y/r & z/r \end{vmatrix}$  is given by  $\frac{\partial}{\partial z} \frac{y}{r} - \frac{\partial}{\partial y} \frac{z}{r} = -\frac{yz}{r^3} + \frac{yz}{r^3} = 0$ . From symmetry w.r.t. cyclic permutation, we obtain  $\nabla \times (\mathbf{r}/r) = 0$ .

Alternatively, with subscript notation:  $\nabla \times \frac{\mathbf{r}}{r} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{x_k}{r} = \varepsilon_{ijk} \frac{\delta_{jk} r - x_k x_j / r}{r^2}$

$$= \mathbf{e}_i \frac{\varepsilon_{ijj} r - \varepsilon_{ijk} x_k x_j / r}{r^2} = -\frac{\varepsilon_{ijk} x_j x_k}{r^3} = -\frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0, \text{ since } \varepsilon_{ijj} = 0, \varepsilon_{ijk} x_j x_k = \mathbf{r} \times \mathbf{r} \text{ and } \mathbf{u} \times \mathbf{u} = 0$$

for any vector  $\mathbf{u}$ .

$$(c) \quad \text{The } x\text{-component of } \nabla \times (\mathbf{r}/r^3) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x/r^3 & y/r^3 & z/r^3 \end{vmatrix} \text{ is given by } \frac{\partial}{\partial z} \frac{y}{r^3} - \frac{\partial}{\partial y} \frac{z}{r^3} = -\frac{3yz}{r^5} +$$

$$3\frac{yz}{r^5} = 0. \text{ From symmetry w.r.t. cyclic permutation, we obtain } \nabla \times (\mathbf{r}/r^3) = 0.$$

Alternatively, with subscript notation:  $\nabla \times (\mathbf{r}/r^3) = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \frac{x_k}{r^3} = \varepsilon_{ijk} \frac{\delta_{jk} r^3 - x_k 3r^2 x_j / r}{r^6} =$

$$\frac{\varepsilon_{ijj} r^3 - 3\varepsilon_{ijk} x_k x_j r}{r^6} = -3\frac{\varepsilon_{ijk} x_j x_k}{r^5} = -3\frac{\mathbf{r} \times \mathbf{r}}{r^5} = 0, \text{ since } \varepsilon_{ijj} = 0, \varepsilon_{ijk} x_j x_k = \mathbf{r} \times \mathbf{r} \text{ and } \mathbf{u} \times \mathbf{u} = 0$$

for any vector  $\mathbf{u}$ .

**Solution 9**

A vector field  $\mathbf{v}$  is irrotational if  $\nabla \times \mathbf{v} = \mathbf{0}$ . Then  $\nabla \times (2y, 2x + g(z), 3yz^2) = \left(3z^2 - \frac{dg}{dz}, 0, 0\right) = 0$  if  $\frac{dg}{dz} = 3z^2 \Rightarrow g = z^3 + C$ , where  $C$  is an arbitrary constant.

**Solution 10**

- (a)  $\nabla \times (2xyz, x^2z, x^2y) = (0, 0, 0) \Rightarrow$  irrotational.  
 (b)  $\nabla \times (0, 1, xy) = (x, -y, 0) \neq 0 \Rightarrow$  not irrotational.

**Solution 11**

- (a)  $\nabla \cdot (2xyz, x^2z, x^2y) = 2yz \Rightarrow$  not solenoidal.  
 (b)  $\nabla \cdot (0, 1, xy) = 0 \Rightarrow$  solenoidal.

**Solution 12**

Using the chain rule,  $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = \mathbf{v} \cdot \nabla f$ .

**Solution 13**

All the expressions below are symmetric with respect to cyclic permutations,  $x \rightarrow y, y \rightarrow z, z \rightarrow x$ . Therefore, it is sufficient to calculate just one component of the vector identity. (But the solution given below includes all three components for illustrative purposes.)

We use the following relation:  $\frac{\partial f(r)}{\partial x} = f' \frac{x}{r}$ , where  $f' = \frac{df}{dr}$ , and similarly for  $y$  and  $z$ .

$$\begin{aligned} \mathbf{c} \times f(r)\mathbf{r} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ c_x & c_y & c_z \\ f_x & f_y & f_z \end{vmatrix} = (c_y f_z - c_z f_y, c_z f_x - c_x f_z, c_x f_y - c_y f_x). \\ \nabla \times (\mathbf{c} \times f(r)\mathbf{r}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(c_y z - c_z y) & f(c_z x - c_x z) & f(c_x y - c_y x) \end{vmatrix} \\ &= \hat{\mathbf{x}} \left[ \frac{\partial}{\partial y} f(c_x y - c_y x) - \frac{\partial}{\partial z} f(c_z x - c_x z) \right] \\ &\quad + \hat{\mathbf{y}} \left[ \frac{\partial}{\partial z} f(c_y z - c_z y) - \frac{\partial}{\partial x} f(c_x y - c_y x) \right] \\ &\quad + \hat{\mathbf{z}} \left[ \frac{\partial}{\partial x} f(c_z x - c_x z) - \frac{\partial}{\partial y} f(c_y z - c_z y) \right] \\ &= \hat{\mathbf{x}} \left[ f' \frac{y}{r} (c_x y - c_y x) + f c_x - f' \frac{z}{r} (c_z x - c_x z) + f c_x \right] \\ &\quad + \hat{\mathbf{y}} \left[ f' \frac{z}{r} (c_y z - c_z y) + f c_y - f' \frac{x}{r} (c_x y - c_y x) + f c_y \right] \\ &\quad + \hat{\mathbf{z}} \left[ f' \frac{x}{r} (c_z x - c_x z) + f c_z - f' \frac{y}{r} (c_y z - c_z y) + f c_z \right] \\ &= 2f\mathbf{c} + \frac{f'}{r} \left[ \hat{\mathbf{x}} (c_x y^2 - c_y x y - c_z x z + c_x z^2) \right. \\ &\quad \left. + \hat{\mathbf{y}} (c_y z^2 - c_z y z - c_x x y + c_y x^2) \right. \\ &\quad \left. + \hat{\mathbf{z}} (c_z x^2 - c_x x z - c_y y z + c_z y^2) \right] \end{aligned}$$

$$\begin{aligned}
&= 2f\mathbf{c} + \frac{f'}{r} \left\{ \hat{\mathbf{x}} \left[ c_x(x^2 + y^2 + z^2) - c_x x^2 - c_y xy - c_z xz \right] \right. \\
&\quad \left. + \hat{\mathbf{y}} \left[ c_y(x^2 + y^2 + z^2) - c_y y^2 - c_z yz - c_x xy \right] \right. \\
&\quad \left. + \hat{\mathbf{z}} \left[ c_z(x^2 + y^2 + z^2) - c_z z^2 - c_x xz - c_y yz \right] \right\} \\
&= 2f\mathbf{c} + \frac{f'}{r} \mathbf{c} r^2 - \frac{f'}{r} [\hat{\mathbf{x}}x(\mathbf{c} \cdot \mathbf{r}) + \hat{\mathbf{y}}y(\mathbf{c} \cdot \mathbf{r}) + \hat{\mathbf{z}}z(\mathbf{c} \cdot \mathbf{r})] \\
&= 2f(r)\mathbf{c} + \frac{1}{r} \frac{df}{dr} [cr^2 - \mathbf{r}(\mathbf{c} \cdot \mathbf{r})] .
\end{aligned}$$

Otherwise, using the subscript notation, consider the  $i$ -th component of the identity:

$$[\nabla \times (\mathbf{c} \times f(r)\mathbf{r})]_i = \varepsilon_{ijk} \partial_j [\mathbf{c} \times f(r)\mathbf{r}]_k .$$

To find  $[\mathbf{c} \times f(r)\mathbf{r}]_k$ , we use the definition  $[\mathbf{a} \times \mathbf{b}]_i = \varepsilon_{ijk} a_j b_k$ , where we replace  $i$  by  $k$  and  $j, k$  by  $p, q$ :

$$[\mathbf{c} \times f(r)\mathbf{r}]_k = \varepsilon_{kpq} c_p f(r) x_q .$$

Therefore,

$$\begin{aligned}
[\nabla \times (\mathbf{c} \times f(r)\mathbf{r})]_i &= \varepsilon_{ijk} \varepsilon_{kpq} \partial_j [c_p f(r) x_q] \\
&= \varepsilon_{ijk} \varepsilon_{kpq} c_p \partial_j [f(r) x_q] \quad (\text{since } \mathbf{c} \text{ is a CONSTANT vector}) \\
&= \varepsilon_{ijk} \varepsilon_{kpq} c_p \left( f' \frac{x_j}{r} x_q + f \delta_{jq} \right) ,
\end{aligned}$$

since  $\frac{\partial}{\partial x_j} f(r) = \frac{df}{dr} \frac{\partial r}{\partial x_j} = f' \frac{x_j}{r}$ , where  $f' = \frac{df}{dr}$ .

To evaluate  $\varepsilon_{ijk} \varepsilon_{kpq}$ , we use the identity  $\varepsilon_{ijk} \varepsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$ . However, summation is carried over the FIRST index in this identity, but the index  $k$ , over which summation is understood in  $\varepsilon_{ijk} \varepsilon_{kpq}$ , is not the first one. So, we should first transform the expression to be evaluated in order to make  $k$  the first index, and only then the above identity can be applied after suitably renaming the indices:

$$\varepsilon_{ijk} \varepsilon_{kpq} = -\varepsilon_{ikj} \varepsilon_{kpq} = \varepsilon_{kij} \varepsilon_{kpq} = \delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp} ,$$

so that we obtain

$$\begin{aligned}
[\nabla \times (\mathbf{c} \times f(r)\mathbf{r})]_i &= (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) c_p \left( f' \frac{x_j}{r} x_q + f \delta_{jq} \right) \\
&= \delta_{ip} \delta_{jq} \left( f' \frac{x_j}{r} x_q + f \delta_{jq} \right) - \delta_{iq} \delta_{jp} \left( f' \frac{x_j}{r} x_q + f \delta_{jq} \right) .
\end{aligned}$$

Now note that, say,  $\delta_{ip} c_p = c_i$ ,  $\delta_{jq} x_q = x_j$ , that  $\delta_{jq} \delta_{jq} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ , and that  $\delta_{iq} \delta_{jp} \delta_{jq} = \delta_{iq} \delta_{pq} = \delta_{ip}$  to obtain

$$\begin{aligned}
[\nabla \times (\mathbf{c} \times f(r)\mathbf{r})]_i &= f' c_i \frac{x_j x_j}{r} + 3c_i f \\
&\quad - f' c_j \frac{x_j x_i}{r} - c_i \\
&= \frac{1}{r} f' c_i r^2 + 2c_i f - f' \frac{x_i}{r} (\mathbf{r} \cdot \mathbf{c}) ,
\end{aligned}$$

which represents the  $i$ -th component of the identity to be proved. Here  $f' = df/dr$ .