MAS2104/MAS3104 **Problems, Part 3**

Surfaces and Normal Vectors. The Gradient

Question 1

Express equations of the following surfaces in the forms **(a)** $f(x, y, z) = 0$ and **(b)** $z = g(x, y)$, and, using both these forms, find a normal vector; also find the corresponding unit normal vector. In each case, sketch the surface.

(i) $(u, v) \mapsto (v \cos u, v \sin u, \sqrt{1 + v^2}),\ 0 \le u \le 2\pi,\ 0 \le v \le 1$,

(ii) $(u, v) \mapsto (v \sin u, v \cos u, \sqrt{v^2 - 1}),\ 0 \le u \le 2\pi,\ 1 \le v \le 2$.

Question 2

Find the points of intersection of the line $t \mapsto (t, t, 2t),\ -\infty < t < \infty$, and the cylinder $(u, v) \mapsto (\sin u, \cos u, v),\ 0 \le u \le 2\pi,\ -\infty < v < \infty$. Find the angle between the tangent to the curve and the normal to the surface at these points.

Question 3

Find a normal to the surface $x = y^2 + z^2$ at the point $(2, 1, -1)$. Find also a unit normal. Is there any other unit normal vector?

Question 4

Find ∇f for **(a)** $f = yz$, **(b)** $f = x^2yz$, **(c)** $f = e^x y + z^3$, **(d)** $f = \sin x + yz$.

Question 5

Consider a scalar field $f = x^3 + y^3 + z^3 - 3xyz$. Find all the points at which ∇f is parallel to the z-axis.

Question 6

Find **(a)** ∇r , **(b)** $\nabla \dfrac{1}{r}$, where $r = |\mathbf{r}|$, $\mathbf{r} = (x, y, z)$. Express your answers in terms of $\mathbf{r}$.

Solutions

Solution 1

(i) $\begin{cases} x = v\cos u\,, \\ y = v\sin u\,, \\ z = \sqrt{1+v^2}\,, \end{cases} \quad \Rightarrow \quad x^2+y^2 = v^2(\sin^2 u + \cos^2 u) = v^2 = z^2 - 1\,.$

I.e. an equation of the surface is $x^2+y^2 = z^2-1$. This equation can be rearranged into form (a) as $x^2+y^2-z^2+1=0$, and into form (b) as $z=\sqrt{x^2+y^2+1}$.

A normal is most easily found using the form (a), where it is given by the gradient of $f(x,y,z)$:

$$\mathbf{n} = \nabla f \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x}f + \hat{\mathbf{y}}\frac{\partial}{\partial y}f + \hat{\mathbf{z}}\frac{\partial}{\partial z}f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Thus, a normal vector obtained from form (a) for this surface is $\mathbf{n} = (2x, 2y, -2z)$ and $\hat{\mathbf{n}} = \dfrac{2(x,y,-z)}{2\sqrt{x^2+y^2+z^2}} = \dfrac{(x,y,-z)}{\sqrt{2z^2-1}}$.

A normal vector can be also found using the form (b) as

$$\mathbf{n} = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right).$$

This formula is easy to remember if we rewrite the form (b) as $f_1(x,y,z) = z - g(x,y) = 0$ and apply $\mathbf{n} = \nabla f_1$. (Note that the function f_1 obtained in this way need not be identical to the function f obtained another way. The equation of a surface in the form $f=0$ is not unique; we can multiply f by any scalar and still satisfy this equation.) For the surface here, using the formula above gives $\mathbf{n} = \left(-\dfrac{x}{z}, -\dfrac{y}{z}, 1\right) = -\dfrac{1}{z}(x,y,-z)$ and $\hat{\mathbf{n}} = -\dfrac{(x,y,-z)}{\sqrt{x^2+y^2+z^2}} = -\dfrac{(x,y,-z)}{\sqrt{2z^2-1}}$. This vector is ANTIPARALLEL to the normal vector obtained using the form (a) above, but both are normal to the surface.

The horizontal cross-sections of this surface are circles $x^2+y^2=z^2-1$, and the surface occupies a region with $z^2 \geq 1$, or $z \geq 1$ (since $z = \sqrt{1+v^2} \geq 0$, so we choose the positive branch of the square root). Since any horizontal cross-section of the surface is a circle centred at the z-axis, the whole surface is rotationally symmetric with respect to the z-axis. We say that this is a surface of revolution as it is swept by a certain curve rotated around the z-axis. The shape of this curve can be understood by considering vertical cross-sections of the surface as follows. For $x=0$ and $y=0$, we have $z=\sqrt{y^2+1}$ and $z=\sqrt{x^2+1}$, respectively, so the vertical cross-sections are curves passing through the point $(0,0,1)$ and asymptotically approaching the lines $z=|x|$ and $z=|y|$ as $|x|,\ |y| \to \infty$. This surface is called a *hyperboloid*, and it is shown in Fig. 1(i) (more precisely, this is the upper sheet of a hyperboloid of two sheets).

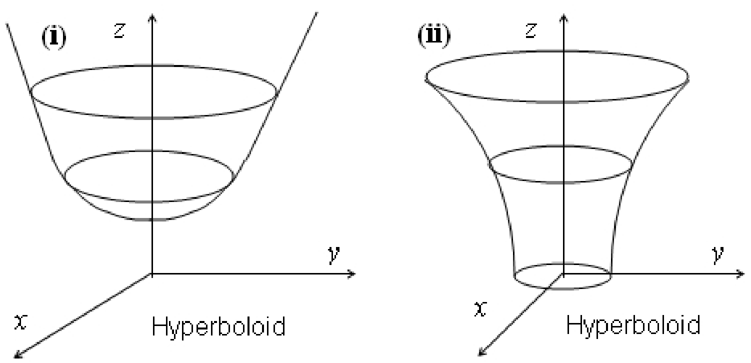

Figure 1: Surfaces of Question 1.

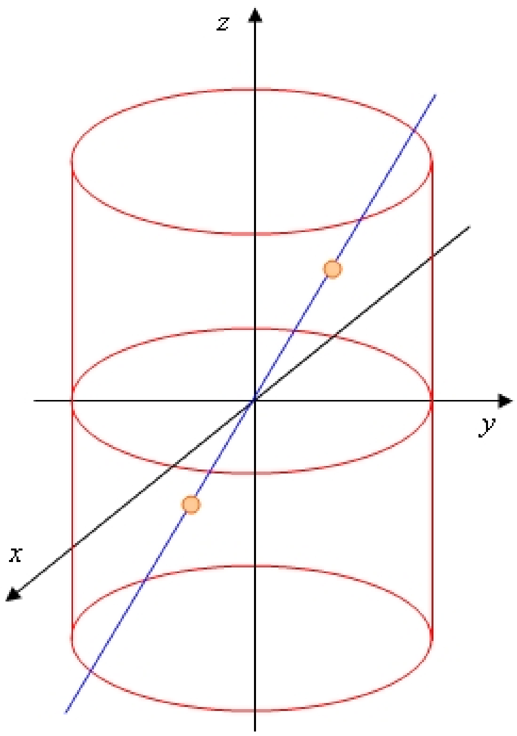

Figure 2: The straight line and cylinder of Question 2 intersect at two points.

(ii) $\begin{cases} x = v\sin u\,, \\ y = v\cos u\,, \\ z = \sqrt{v^2-1}\,, \end{cases} \quad\Rightarrow\quad x^2+y^2 = v^2(\sin^2 u + \cos^2 u) = v^2 = z^2+1\,.$

Thus, the form (a) is $x^2+y^2-z^2-1=0$, and the form (b) is $z=\sqrt{x^2+y^2-1}$ — note that $z \geq 0$ as above.

A normal obtained from the form (a) is $\mathbf{n} = \nabla f = 2(x, y, -z)$ and $\widehat{\mathbf{n}} = \dfrac{(x, y, -z)}{\sqrt{x^2+y^2+z^2}} = \dfrac{(x, y, -z)}{\sqrt{2z^2+1}}$.

For the form (b), we have $\mathbf{n} = \left(-\dfrac{x}{z}, -\dfrac{y}{z}, 1\right) = -\dfrac{1}{z}(x, y, -z)$ and $\widehat{\mathbf{n}} = -\dfrac{(x, y, -z)}{\sqrt{x^2+y^2+z^2}} = -\dfrac{(x, y, -z)}{\sqrt{2z^2+1}}$.

The horizontal cross-sections of this surface are circles $x^2+y^2=z^2+1$, so this is a surface of revolution. The radius of the horizontal cross-section has the minimal value 1 at $z=0$. For $x=0$ and $y=0$, we have $z=\sqrt{y^2-1}$ and $z=\sqrt{x^2-1}$, respectively, so these two vertical cross-sections are curves passing through the points $(0,1,0)$ and $(1,0,0)$ and the surface is not defined at $|x|, |y| < 1$. As in the previous example, the vertical cross-sections approach asymptotically the lines $z=|x|$ and $z=|y|$ as $|x|, |y| \to \infty$. This surface is also called a *hyperboloid*, and it is shown in Fig. 1(ii).

The above examples make clear that a normal to any surface of revolution symmetric with respect to the z-axis has its horizontal components parallel to the radial direction in the (x, y)-plane. It is instructive to compare equations of and normals to the surfaces of Parts (i) and (ii): these are two closely related surfaces, hyperboloids of two and one sheet, respectively.

Solution 2

It is useful to start with a sketch of the curve (which is in fact a straight line $y=x,\ z=2x$) and the surface (which is a circular cylinder symmetric with respect to the z-axis), as shown in Fig. 2.

It is immediately clear that the straight line and the cylinder intersect at two points.

For the points belonging to both the line and the surface, we have the following three simultaneous equations for t, u and v:

$$\begin{cases} t = \sin u \, , \\ t = \cos u \, , \\ 2t = v \, , \end{cases} \quad \Rightarrow \quad \sin u = \cos u \quad \Rightarrow \quad \tan u = 1 \quad \Rightarrow \quad u = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

(note that $0 \le u \le 2\pi$, so these are the only admissible solutions). From u we can find the value of v at the intersection points using $v = 2t = 2\sin u$. For $u = \pi/4$, the intersection point is $(x, y, z) = (\sin u, \cos u, 2\sin u) = (1/\sqrt{2}, 1/\sqrt{2}, \sqrt{2})$. The other intersection point has $u = 5\pi/4$, so it is $(x, y, z) = (\sin u, \cos u, 2\sin u) = (-1/\sqrt{2}, -1/\sqrt{2}, -\sqrt{2})$.

Now we have to find the tangent to the line and the normal to the surface at the above two points. The tangent to the line is given by $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z}) = (1, 1, 2)$. The curve is a straight line, so the tangent is a constant vector.

The equation of the surface can be rewritten as $f(x, y, z) = x^2 + y^2 - 1$, so its normal is $\mathbf{n} = \nabla f = (2x, 2y, 0)$. Alternatively, see the *Lecture Notes* for the derivation of a normal to a cylinder given in the above parametric form. So, at the two points of intersection we have $\mathbf{n} = (2x, 2y, 0) = (\sqrt{2}, \sqrt{2}, 0)$ and $(-\sqrt{2}, -\sqrt{2}, 0)$, respectively.

The dot product of $\mathbf{v}$ and $\mathbf{n}$ at the two intersection points is $\mathbf{v} \cdot \mathbf{n} = \pm 2\sqrt{2}$.

The moduli of the two vectors are $|\mathbf{v}| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$ and $|\mathbf{n}| = \sqrt{(2x)^2 + (2y)^2} = \sqrt{[2/(\pm\sqrt{2})]^2 + [2/(\pm\sqrt{2})]^2} = \sqrt{2+2} = 2$.

Hence, cosine of the angle between the two vectors is $\cos\theta = \pm 2\sqrt{2}/(2\sqrt{6}) = \pm 1/\sqrt{3}$. So, $\theta = \arccos(1/\sqrt{3})$ and $\theta = \arccos(-1/\sqrt{3}) = \pi - \arccos(1/\sqrt{3})$ at the two points of intersection.

Solution 3

A form of the surface equation especially convenient to calculate a normal vector is $x - y^2 - z^2 = 0$, so $\mathbf{n} = \nabla(x - y^2 - z^2) = (1, -2y, -2z) = (1, -2, 2)$ at $(x, y, z) = (2, 1, -1)$. Since $|\mathbf{n}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$, the unit normal is $\widehat{\mathbf{n}} = \dfrac{\mathbf{n}}{|\mathbf{n}|} = \dfrac{1}{3}(1, -2, 2)$.

The only other unit normal vector is $\widehat{\mathbf{n}}' = -\widehat{\mathbf{n}} = \dfrac{1}{3}(-1, 2, -2)$.

Solution 4

(a) $\nabla f = (0, z, y)$. **(b)** $\nabla f = (2xyz, x^2 z, x^2 y)$. **(c)** $\nabla f = (e^x y, e^x, 3z^2)$.
(d) $\nabla f = (\cos x, z, y)$.

Solution 5

First, we evaluate the gradient: $\nabla f = (3x^2 - 3yz)\widehat{\mathbf{x}} + (3y^2 - 3xz)\widehat{\mathbf{y}} + (3z^2 - 3xy)\widehat{\mathbf{z}}$.

If ∇f is parallel to the z-axis, its x- and y-components must be equal to zero. Thus, we have the following two simultaneous equations defining these points:

$$\begin{cases} x^2 - yz = 0 \, , \\ y^2 - xz = 0 \, . \end{cases} \quad \text{Assuming that } y \neq 0,\ x \neq 0, \text{ this yields } \begin{cases} z = x^2/y \\ z = y^2/x \end{cases}$$

$$\Rightarrow \quad \frac{x^2}{y} = \frac{y^2}{x} \quad \Rightarrow \quad x^3 = y^3 \quad \Rightarrow \quad x = y \, .$$

Then $z = x^2/y = x$, so that we can conclude that $\nabla f \parallel \widehat{\mathbf{z}}$ on a straight line $x = y = z$.

In addition, $x = y = z = 0$ is also a solution of the above equations (before we have divided them by x and y). However, $\nabla f = 0$ at the origin, and the zero vector cannot be parallel to anything, so $(x, y, z) = (0, 0, 0)$ is not a solution.

Solution 6

(a) $$\nabla r = \nabla\sqrt{x^2+y^2+z^2} = \widehat{\mathbf{x}}\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} + \widehat{\mathbf{y}}\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2} + \widehat{\mathbf{z}}\frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}$$
$$= \widehat{\mathbf{x}}\frac{x}{\sqrt{x^2+y^2+z^2}} + \widehat{\mathbf{y}}\frac{y}{\sqrt{x^2+y^2+z^2}} + \widehat{\mathbf{z}}\frac{z}{\sqrt{x^2+y^2+z^2}} = \widehat{\mathbf{x}}\frac{x}{r} + \widehat{\mathbf{y}}\frac{y}{r} + \widehat{\mathbf{z}}\frac{z}{r}$$
$= (1/r)(x\widehat{\mathbf{x}} + y\widehat{\mathbf{y}} + z\widehat{\mathbf{z}}) = \mathbf{r}/r \equiv \widehat{\mathbf{r}}$, where $\widehat{\mathbf{r}}$ is the unit radius-vector.

NB There is actually no need to perform the calculations in full for each of the three components of ∇r, since the components are related to each other by symmetry. (The full calculations are presented here largely for illustrative purposes, if this type of symmetry is not yet clear.) It is more efficient to calculate, say, $\partial r/\partial x$, and then deduce the expressions for $\partial r/\partial y$ and $\partial r/\partial z$ from the symmetry of both r and ∇ with respect to x, y and z.

If we were using the optional Subscript Notation (see the Handout on this, to be distributed shortly), we could write $r = \sqrt{x_j x_j}$ and $\nabla r = \mathbf{e}_i \frac{\partial r}{\partial x_i} = \mathbf{e}_i \frac{\partial}{\partial x_i}\sqrt{x_j x_j}$. Note that we use index j rather than i for the INDEPENDENT sum under the square root. Thus,

$$\nabla r = \mathbf{e}_i \tfrac{1}{2}\frac{2x_j \frac{\partial x_j}{\partial x_i}}{\sqrt{x_j^2}} = \mathbf{e}_i \frac{x_j \delta_{ij}}{r} = \frac{\mathbf{e}_i x_i}{r} = \frac{\mathbf{r}}{r} = \widehat{\mathbf{r}} \ ,$$

where we have used $\frac{\partial x_j}{\partial x_i} = \delta_{ij}$ and $x_j \delta_{ij} = x_i$.

(b) Consider first $\frac{\partial}{\partial x}\frac{1}{r}$. This is equal to $-\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{x}{r} = -\frac{x}{r^3}$.

Since x, y and z enter both r and ∇ in a completely symmetric manner, we can conclude without any additional calculation that $\frac{\partial}{\partial y}\frac{1}{r} = -\frac{y}{r^3}$ and $\frac{\partial}{\partial z}\frac{1}{r} = -\frac{z}{r^3}$.

Hence, $\nabla\frac{1}{r} = -\frac{x}{r^3}\widehat{\mathbf{x}} - \frac{y}{r^3}\widehat{\mathbf{y}} - \frac{z}{r^3}\widehat{\mathbf{z}} = -\frac{\mathbf{r}}{r^3}$.

Alternatively, $\nabla\frac{1}{r} = -\frac{1}{r^2}\nabla r = -\frac{1}{r^2}\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{r^3}$ as above, but with less effort.

Or, if using the subscript notation: $\nabla r^{-1} = \mathbf{e}_i \frac{\partial}{\partial x_i} r^{-1} = -\mathbf{e}_i r^{-2}\frac{\partial r}{\partial x_i} = -\mathbf{e}_i \frac{x_i}{r^3} = -\frac{\mathbf{r}}{r^3}$.