Curves in Two and Three Dimensions. Families of Curves

Question 1

Find the length of the following curves [cf. Question 4(a) of Problems, Part 1]:

$$\text{(a)}\ t \mapsto (t, e^t)\ ,\ -1 \le t \le 1\ ; \quad \text{(b)}\ t \mapsto (\tfrac{1}{2}t^2, \tfrac{1}{3}t^3)\ ,\ 0 \le t \le 1\ .$$

Hint: to evaluate the integral in (a), $\int_{-1}^{1} \sqrt{1+\exp(2t)}\, dt$, use a new variable u defined via $u^2 = 1 + \exp(2t)$.

Question 2

For the curve given in Question 1(b), find another parametric form with distance s along the curve as the curve parameter; indicate clearly the range of s.

Hint: Evaluate $s(t) = \int_{t_1}^{t} |\mathbf{v}|\, dt'$ and then express t in terms of s, i.e., find $t = t(s)$, and substitute the result for t in the equation on the curve.

Question 3

Sketch the following one-parameter family of curves, indicating clearly the region covered by the family and the directions in which the curves are traversed at t increases:

$$t \to (t, \lambda e^t)\ ,\ 0 \le t \le 1\ ,\ -1 \le \lambda \le 1\ .$$

Question 4

Find where the following two curves intersect (you may find Handout 1 useful), and find the angle between their tangents at these points:

$$t \mapsto (\cos t, \sin t, t),\ -\infty < t < \infty \qquad \text{and} \qquad t' \mapsto (\sin t', \cos t', t'),\ -\infty < t' < \infty\ .$$

Question 5

Consider two curves, $t \mapsto (\cos t, \sin t, t)$, $-\infty < t < \infty$, and $t' \mapsto (t', t', at')$, $-\infty < t' < \infty$, where a is a constant. Find those values of a for which the two curves intersect. Make a sketch showing the intersection point for two distinct values of a found above and for a value of a where the curves do not intersect. Find out whether the two curves can intersect at more than one point.

Question 6

Sketch the family of curves

$$t \mapsto (\lambda \cos t, \lambda \sin t, \lambda),\ 0 \le t \le 2\pi,\ -1 \le \lambda \le 1\ ,$$

where t is in radians. Indicate with arrows the directions in which the curves are traversed at t increases, and show the points or curves corresponding to the extereme values of t and λ. State whether the family covers up a surface or fills in a volume and give the name of this surface or solid.

Solutions

Solution 1

The length of a curve

$$b : t \mapsto \Big(x(t), y(t)\Big),\ t_1 \le t \le t_2 ,$$

is given by

$$S = \int_{t_1}^{t_2} \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt = \int_{t_1}^{t_2} |\mathbf{v}|\, dt .$$

(a) Here $\mathbf{v} = (\dot{x}, \dot{y}) = (1, e^t) \quad \Rightarrow \quad S = \int_{-1}^{1} \sqrt{1 + e^{2t}}\, dt.$

Let $u^2 = 1 + e^{2t}$. Then $2u\, du = 2e^{2t}\, dt \quad \Rightarrow \quad dt = \dfrac{u}{\exp(2t)}\, du = \dfrac{u}{u^2 - 1}\, du$. Thus,

$$\begin{aligned} S &= \int_{\sqrt{1+e^{-2}}}^{\sqrt{1+e^2}} u \frac{u}{u^2-1}\, du = \int_{\sqrt{1+e^{-2}}}^{\sqrt{1+e^2}} \frac{u^2 - 1 + 1}{u^2 - 1}\, du \\ &= \int_{\sqrt{1+e^{-2}}}^{\sqrt{1+e^2}} \left(1 + \frac{1}{u^2-1}\right) du = u\Big|_{\sqrt{1+e^{-2}}}^{\sqrt{1+e^2}} + \tfrac{1}{2} \int_{\sqrt{1+e^{-2}}}^{\sqrt{1+e^2}} \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du \\ &= \sqrt{1+e^2} - \sqrt{1+e^{-2}} + \tfrac{1}{2} \ln \frac{u-1}{u+1}\Big|_{\sqrt{1+e^{-2}}}^{\sqrt{1+e^2}} \\ &= \sqrt{1+e^2} - \sqrt{1+e^{-2}} + \tfrac{1}{2} \ln \frac{\left(\sqrt{1+e^2} - 1\right)\left(\sqrt{1+e^{-2}} + 1\right)}{\left(\sqrt{1+e^2} + 1\right)\left(\sqrt{1+e^{-2}} - 1\right)}. \end{aligned}$$

(b) Now, $\mathbf{v} = (\dot{x}, \dot{y}) = (t, t^2) \quad \Rightarrow \quad |\mathbf{v}| = t\sqrt{1+t^2} \quad \Rightarrow \quad S = \int_0^1 t\sqrt{1+t^2}\, dt$.

Introduce $u = 1 + t^2,\ du = 2t\, dt$ to obtain

$$S = \tfrac{1}{2} \int_1^2 u^{1/2}\, du = \tfrac{1}{2}\, \tfrac{2}{3} u^{3/2}\Big|_1^2 = \tfrac{1}{3}\left(2\sqrt{2} - 1\right).$$

Solution 2

The distance s along the curve $b : t \mapsto \Big(x(t), y(t)\Big),\ t_1 \le t \le t_2$ is given by

$$s(t) = \int_{t_1}^{t} \sqrt{(\dot{x})^2 + (\dot{y})^2}\, dt' \equiv \int_{t_1}^{t} |\mathbf{v}(t')|\, dt' ,$$

where the dummy integration variable is denoted t' to distinguish it from t.

To solve the problem, we shall find s as a function of t by evaluating the above integral, and then use $s = s(t)$ to express t as a function of s, that is to find $t = t(s)$, and then we shall use the result to express $x(t)$ and $y(t)$ as functions of s.

For the curve

$$b : \quad t \to (\tfrac{1}{2}t^2, \tfrac{1}{3}t^3), \quad 0 \le t \le 1 ,$$

$$\mathbf{v} = (\dot{x}, \dot{y}) = (t, t^2) \quad \Rightarrow \quad |\mathbf{v}| = t\sqrt{1+t^2} \quad \Rightarrow \quad s(t) = \int_0^t t' \sqrt{1 + t'^2}\, dt' .$$

Introduce $u = 1 + t'^2,\ du = 2t'\, dt'$ to obtain

$$s(t) = \tfrac{1}{2} \int_1^{1+t^2} u^{1/2}\, du = \tfrac{1}{2}\, \tfrac{2}{3} u^{3/2}\Big|_1^{1+t^2} = \tfrac{1}{3}\left[(1+t^2)^{3/2} - 1\right].$$

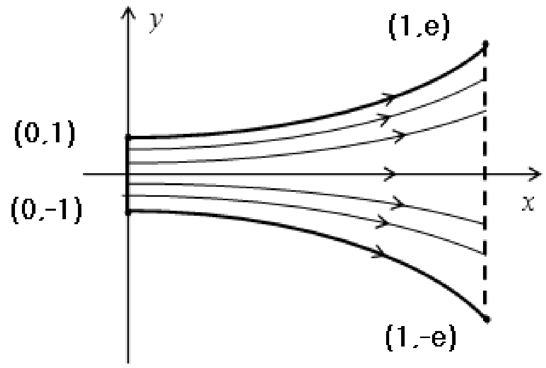

Figure 1: The family of curves of Question 3.

We check that $s(0) = 0$ and that the maximum value of $s(t)$ is the total length of the curve, $S = s(1) = \frac{1}{3}(2\sqrt{2} - 1)$ as obtained in Question 1(b).

Now we have to express t in terms of s:

$$(1 + t^2)^{3/2} = 3s + 1 \quad \Rightarrow \quad t = \left[(3s + 1)^{2/3} - 1\right]^{1/2} .$$

Thus, the parametric representation of the curve in terms of s is given by

$$s \to \left(\tfrac{1}{2}\left[(3s + 1)^{2/3} - 1\right], \tfrac{1}{3}\left[(3s + 1)^{2/3} - 1\right]^{3/2}\right), \quad 0 \le s \le \tfrac{1}{3}(2\sqrt{2} - 1) .$$

Solution 3

With $x(t, \lambda) = t,\ y(t, \lambda) = \lambda e^t,\ 0 \le t \le 1$, we easily see that the curves of the family are of the form $y = \lambda e^x$ and they are located to the right of the y-axis because $0 \le x \le 1$. Since y, together with λ, can be both negative and positive, $-1 \le \lambda \le 1$, the curves of the family are located symmetrically on both sides of the x-axis. Now we calculate the coordinates of the corner points to obtain the sketch shown in Fig. 1.

Solution 4

The following equations must be satisfied at any intersection point of the curves (because each intersection point belongs to both curves; we assume multiple intersections):

$$\begin{cases} \cos t = \sin t' , \\ \sin t = \cos t' , \\ t = t' , \end{cases} \quad \Rightarrow \quad t = t' \text{ and } \cos t = \sin t \quad \Rightarrow \quad \tan t = 1 \quad \Rightarrow \quad t = \tfrac{1}{4}\pi + \pi n$$

where $n = 0, 1, 2, \ldots$.

Thus, the curves intersect infinitely many times at the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right)$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{9\pi}{4}\right)$, $\ldots$. This conclusion is easy to understand as the curves are helices (cf. *Lecture Notes*) with opposite senses of winding: they intersect once per each rotation around the z-axis.

Now we find the tangent vectors: $\mathbf{v} = (-\sin t, \cos t, 1),\ \mathbf{v}' = (\cos t', -\sin t', 1)$. Their dot product is $\mathbf{v} \cdot \mathbf{v}' = -\sin t \cos t' - \cos t \sin t' + 1 = -2 \sin t \cos t + 1$ (since $t = t'$) $\quad \Rightarrow \quad \mathbf{v} \cdot \mathbf{v}' = -\sin 2t + 1$.

At the intersection points, $t = \frac{1}{4}\pi + \pi n$, we have $\mathbf{v} \cdot \mathbf{v}' = -\sin\left(\frac{1}{2}\pi + 2\pi n\right) + 1 = -\sin \frac{1}{2}\pi + 1 = -1 + 1 = 0$. The dot product of two vectors vanishes only when they are orthogonal to each other (if both differ from zero) because then $\cos\theta = \cos \frac{1}{2}\pi = 0$, where θ is the angle between the vectors. Therefore, the tangent vectors to the two curves are orthogonal to each other at all the intersection points: the curves intersect at right angles.

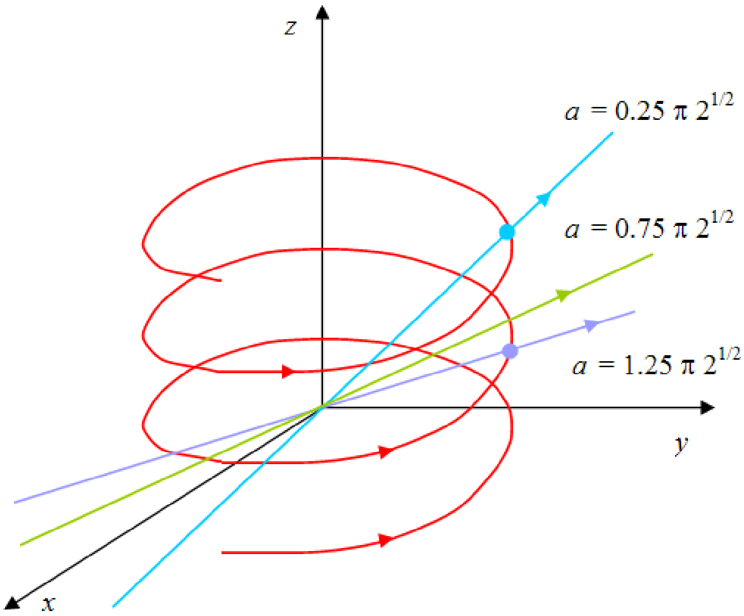

Figure 2: The two curves of Question 5, a helix and a straight line, do or do not intersect depending on the slope of the straight line, a.

Solution 5

For an intersection point, we have the following simultaneous equations:

$$\begin{cases} \cos t = t' \,, \\ \sin t = t' \,, \\ t = at' \,, \end{cases} \quad \Rightarrow \quad \tan t = 1 \quad \Rightarrow \quad t = \tfrac{1}{4}\pi + \pi n, n = 0, 1, \dots \,. \tag{1}$$

Then

$$t' = \sin t = \begin{cases} +1/\sqrt{2} & \text{for } n = 0, 2, \dots \,, \\ -1/\sqrt{2} & \text{for } n = 1, 3, \dots \,, \end{cases} = (-1)^n \frac{1}{\sqrt{2}} \,.$$

Now the third equation of (1) yields

$$\tfrac{1}{4}\pi + \pi n = (-1)^n \frac{a}{\sqrt{2}}$$

and so the curves intersect if a takes any of the following discrete set of values:

$$a = (-1)^n \pi \sqrt{2} \left(\tfrac{1}{4} + n\right) \,. \tag{2}$$

This result is clarified by a sketch of Fig. 2. For the values of a given by Eq. (2), the curves intersect and the value of n determines at what turn of the helix the intersection occurs. If a differs from any of the values of Eq. (2), the curves do not intersect—the straight line passes in between the turns of the helix. There are infinitely many admissible values of a because both curves are infinitely extended.

It is not immediately clear (at least, to me—GRS) whether or not the helix and the straight line can intersect at two points simultaneously, one at $z > 0$ and the other at $z < 0$. To clarify whether this is possible, suppose that there is more than one intersection point. Since Eq. (2) includes ALL points of intersection, there must be two distinct values of n, say n and n', such that they correspond to the same value of a, that is

$$(-1)^n \pi \sqrt{2} \left(\tfrac{1}{4} + n\right) = (-1)^{n'} \pi \sqrt{2} \left(\tfrac{1}{4} + n'\right) \,.$$

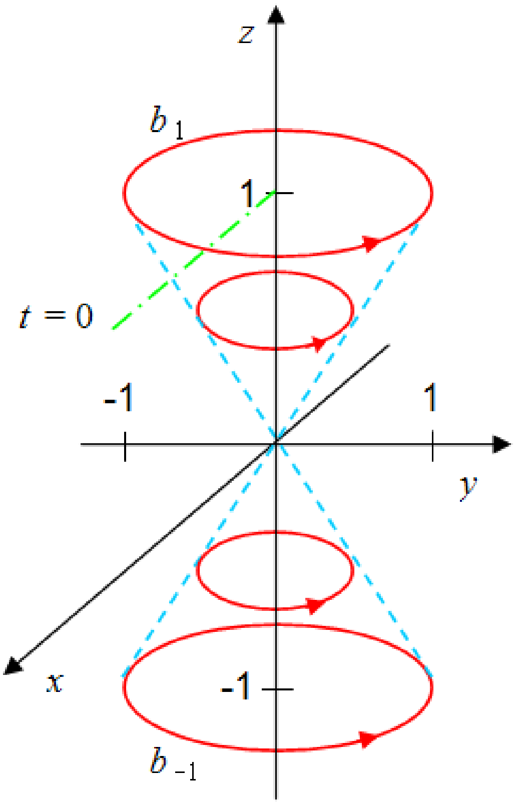

Figure 3: The family of curves of Question 6.

Obviously, this equality may hold with distinct n and n' only if the factors $(-1)^n$ and $(-1)^{n'}$ on the left-hand and right-hand sides have distinct signs, that is one of n and n' is even and the other is odd. Then

$$\tfrac{1}{4} + n = -\tfrac{1}{4} - n' \quad \Rightarrow \quad n + n' = -\tfrac{1}{2} \,,$$

but this cannot be true since both n and n' are integers. Therefore, the curves can only intersect once.

Solution 6

This is a one-parameter family of curves, so it covers up a surface. To understand how the curves look, eliminate t and λ between $x = \lambda \cos t\,,\ y = \lambda \sin t\,,\ z = \lambda$. We first obtain $x^2 + y^2 = \lambda^2$ and then note that $\lambda = z$, so that $x^2 + y^2 = z^2$.

It is clear that each member of the family is a circle of a radius z located at a height z above the (x, y)-plane and parallel to that plane. This is a full circle since t covers a whole period of sine and cosine, $0 \le t \le 2\pi$. The circle radius is 0 at $z = 0$ and it increases linearly with z up to 1 at $|z| = |\lambda| = 1$, the extremal values of z. The points corresponding to $t = 0$ have $y = 0$. The above is perhaps sufficient to conclude that the surface covered up by the family is a circular cone extended from $z = -1$ to $z = 1$ as shown in Fig. 3.

A perhaps more systematic way to sketch a surface given in the form $f(x, y, z) = 0$ (or in any other non-parametric form) is as follows. As an example, consider

$$x^2 + y^2 = z^2 \,, \quad -1 \le z \le 1 \,, \tag{3}$$

and examine various two-dimensional cross-sections of the surface. For example, a horizontal cross-section at a height $z = \lambda$ is given by $x^2 + y^2 = \lambda^2$—this is a circle of a radius λ, and its radius grows LINEARLY with z from 0 at the origin to 1 at $z = \pm 1$.

The vertical cross-section in the (x, z)-plane is obtained by putting $y = 0$ in Eq. (3), which yields $x^2 = z^2$, or $z = \pm x$. This is equation of two straight lines intersecting at the origin. Likewise, the cross-section in the (y, z)-plane is a pair of two straight lines $z = \pm y$ intersecting at the origin. The straight lines extend from $z = -1$ to $z = 1$.

Putting together the horizontal and vertical cross-sections of the surface, we can conclude that this is a cone shown in Fig. 3.