Vectors. Curves in Two Dimensions

Question 1

Find the straight line distance between the points $P_1 = (4, 3)$ and $P_2 = (3, 4)$ in the (x, y) plane.

Question 2

Find the cosine of the angle between the following vectors:
(a) $(\sqrt{2}, 1, -1)$ and $(1, 0, 0)$; **(b)** $(1, 3, \sqrt{6})$ and $(1, 1, 0)$.

Question 3

Consider three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ such that $|\mathbf{a}| = 11$, $|\mathbf{b}| = 23$, and $|\mathbf{a} - \mathbf{b}| = 30$. Find $|\mathbf{a} + \mathbf{b}|$.

Question 4

Sketch the following curves and indicate with an arrow the direction in which the curves are traversed as t increases [t is in radians in (b)]:

(a) $t \mapsto (t, e^t)$, $-1 \le t \le 1$; (b) $t \mapsto (\cos t, t)$, $-\pi \le t \le \pi$;
(c) $t \mapsto (\sqrt{t}, t^2)$, $0 \le t \le 1$.

Question 5

Consider the curves in Question 4. Find the coordinates of the points corresponding to

(a) $t = 0$ for Question 4a ; (b) $t = \frac{1}{4}\pi$ for Question 4b ;
(c) $t = \frac{1}{4}$ for Question 4c .

Question 6

Find tangent vectors $\mathbf{v}$ at the points identified in Question 5.

Question 7

Interpreting t as time and therefore the tangent vector as velocity, find the speeds $v = |\mathbf{v}|$ at the times given in Question 5.

Solutions

Solution 1

For two points in the (x, y)-plane, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the straight line distance between them (which is the shortest distance) is given by $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (as it immediately follows from Pythagoras' Theorem). Thus, $d = \sqrt{(4-3)^2 + (3-4)^2} = \sqrt{1+1} = \sqrt{2}$.

Solution 2

There are two ways to evaluate the dot product of two vectors: one is using the vector components, $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i$, where $\mathbf{a} = (a_1, a_2, a_3)$ and likewise for $\mathbf{b}$. On the other hand the definition of the dot product is

$$\mathbf{a} \cdot \mathbf{b} = ab \cos\theta \ , \tag{1}$$

where θ is the angle between the vectors and $a = |\mathbf{a}| = \sqrt{\sum_{i=1}^{3} a_i^2}$ and similarly for b. Therefore,

$$\cos\theta = \frac{1}{ab} \sum_{i=1}^{3} a_i b_i \ .$$

(a) With $a = \sqrt{(\sqrt{2})^2 + 1^2 + (-1)^2} = 2$ and $b = \sqrt{1^2 + 0^2 + 0^2} = 1$, we obtain
$\cos\theta = \frac{1}{2 \cdot 1}[\sqrt{2} \cdot 1 + 1 \cdot 0 + (-1) \cdot 0] = \frac{\sqrt{2}}{2}$, so $\theta = 45°$.

(b) With $a = \sqrt{1^2 + 3^2 + (\sqrt{6})^2} = 4$ and $b = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$, we obtain
$\cos\theta = \frac{1}{4 \cdot \sqrt{2}}[1 \cdot 1 + 3 \cdot 1 + \sqrt{6} \cdot 0] = \frac{\sqrt{2}}{2}$, so $\theta = 45°$.

NB The dot product can be defined as (1) in a space of any dimension N, and then it can also be calculated as $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{N} a_i b_i$ [note that the latter expression is *not* a definition of the dot product but can rather be obtained from Eq. (1) using the orthogonality of the unit coordinate vectors]. On the contrary, the cross product is specific to the three-dimensional space.

Solution 3

It is useful to note that the length a of a vector $\mathbf{a}$ can be expressed in terms of the dot product of the vector and itself, $a^2 = \mathbf{a} \cdot \mathbf{a}$; this follows directly from the definition of the dot product (1) upon taking $\mathbf{a} = \mathbf{b}$, so that $\theta = 0$. Then

$$|\mathbf{a} - \mathbf{b}|^2 = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2 \ . \tag{2}$$

On the other hand, we can show in a similar manner that $|\mathbf{a} + \mathbf{b}|^2 = a^2 + 2\mathbf{a} \cdot \mathbf{b} + b^2$. In this formula, $a^2 = 11^2$ and $b^2 = 23^3$ are given, and we need to know $2\mathbf{a} \cdot \mathbf{b}$. This can be obtained from Eq. (2) as $2\mathbf{a} \cdot \mathbf{b} = a^2 + b^2 - |\mathbf{a} - \mathbf{b}|^2$, so

$$|\mathbf{a} + \mathbf{b}|^2 = 2a^2 + 2b^2 - |\mathbf{a} - \mathbf{b}|^2 \ .$$

Thus, $|\mathbf{a} + \mathbf{b}|^2 = 2 \cdot 11^2 + 2 \cdot 23^2 - 30^2 = 400$, and so $|\mathbf{a} + \mathbf{b}| = 20$.

Solution 4

This problem can be solved in either of the following two ways:

(i) Make a table of (x, y) values corresponding to various values of t, then plot these points in the (x, y) plane and indicate with an arrow the direction corresponding to increasing t. This method is straightforward and simple, but time consuming. In addition, you may overlook some important features of the curve if you have calculated too few values of x and y.

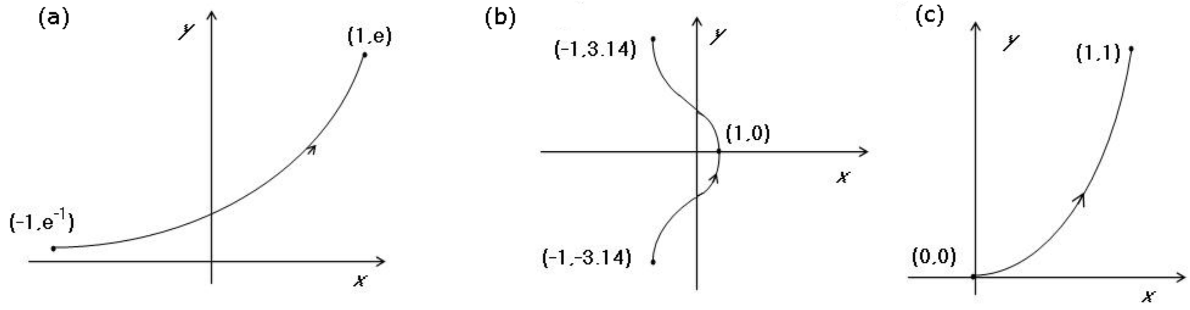

Figure 1: The curves of Question 4.

(ii) Eliminate t between equations $x = x(t)$, $y = y(t)$ to obtain equation of the form $y = y(x)$ [or $x = x(y)$], which often can be plotted easily. Then determine which values of x (or y) correspond to the end points t_1 and t_2. Finally find out which parts of the curve correspond to smaller and larger values of t to identify the direction in which the curve is traversed.

It is always useful to check for any symmetries (such as odd or even in x and/or y), periodic or monotonic behaviour (especially if trigonometric functions appear in the equation), etc. If it is not immediately obvious how to plot the curve, determine its maximum and minimum points (i.e., those where the curve has horizontal tangent, $dy/dx = 0$), points where it crosses the axes ($y = 0$ or $x = 0$, if any), etc. It is also useful to calculate a few values of $x(t)$ and $y(t)$ to check your results. It is also very helpful to determine at which points (if any) the curve has vertical tangent (i.e., where $|dy/dx| \to \infty$).

(a) Here $x(t) = t,\ y(t) = e^t,\ t_1 = -1,\ t_2 = 1$. Eliminating t, we can rewrite this as $y = e^x,\ -1 \le x \le 1$. the curve is shown in Fig. 1a.

(b) With $x(t) = \cos t,\ y(t) = t$, this curve is given by $y = \arccos x$, extended over the interval $-\pi \le y \le \pi$ — see Fig. 1b. Otherwise, you might prefer to consider the curve as a part of $x = \cos y$ in the above interval of y.

(c) Here $x(t) = \sqrt{t},\ y(t) = t^2,\ t_1 = 0,\ t_2 = 1$, so that $y = x^4,\ 0 \le x \le 1$. The sketch is shown in Fig. 1c. Note that the slope is zero at the origin.

Solution 5

Direct calculation leaves no doubt that the answers are
(a) $(0, 1)$; **(b)** $(\sqrt{2}/2, \pi/4)$; **(c)** $(1/2, 1/16)$.

Solution 6

For a curve given in a parametric form, $b : t \to \big(x(t), y(t)\big),\ t_1 \le t \le t_2$, a tangent vector is given by $\mathbf{v} = (\dot{x}, \dot{y})$, where dot denotes derivative with respect to t. Hence,

(a) $x(t) = t,\ y(t) = e^t \quad \Rightarrow \quad \dot{x} = 1,\ \dot{y} = e^t \quad \Rightarrow \quad \mathbf{v} = (1, 1)$ at $t = 0$.

(b) $x(t) = \cos t,\ y(t) = t \quad \Rightarrow \quad \dot{x} = -\sin t,\ \dot{y} = 1 \quad \Rightarrow \quad \mathbf{v} = (-\sqrt{2}/2, 1)$ at $t = \pi/4$.

(c) $x(t) = \sqrt{t},\ y(t) = t^2 \quad \Rightarrow \quad \dot{x} = \frac{1}{2}t^{-1/2},\ \dot{y} = 2t \quad \Rightarrow \quad \mathbf{v} = (1, \frac{1}{2})$ at $t = \frac{1}{4}$.

Solution 7

The speed is the magnitude (modulus) of the velocity vector, i.e., $v = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{(\dot{x})^2 + (\dot{y})^2}$, where $\mathbf{v} \cdot \mathbf{v} = v_x^2 + v_y^2$ and v_x and v_y are the x- and y-components of the vector $\mathbf{v}$, respectively.

Thus, using the components of $\mathbf{v}$ determined in Question 6, we obtain

$$|\mathbf{v}| = \begin{cases} \text{(a)} \;\; \sqrt{2}\,; \\ \text{(b)} \;\; \sqrt{3/2}\,; \\ \text{(c)} \;\; \sqrt{5}/2\,. \end{cases}$$