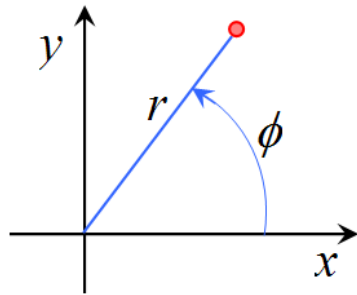


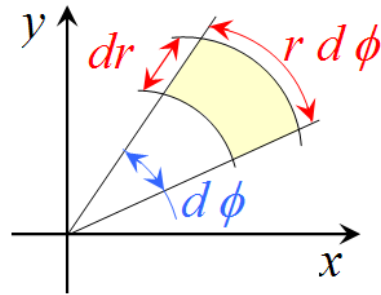
Volume Elements in Cylindrical and Spherical Polar Coordinates

(1) Polar coordinates (r, ϕ) : the area element



$$x = r \cos \phi$$

$$y = r \sin \phi$$



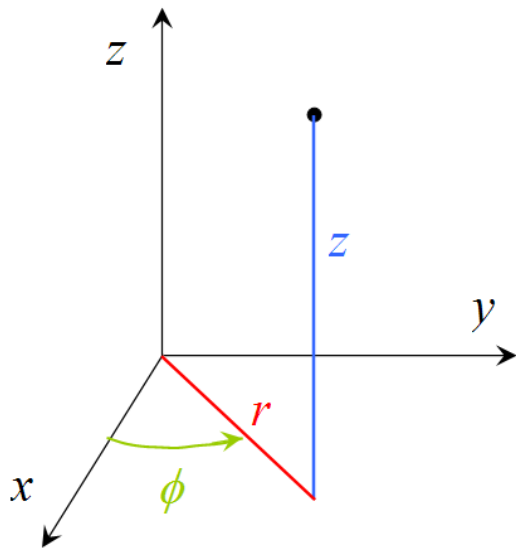
$$dS = r dr d\phi ,$$

the area element

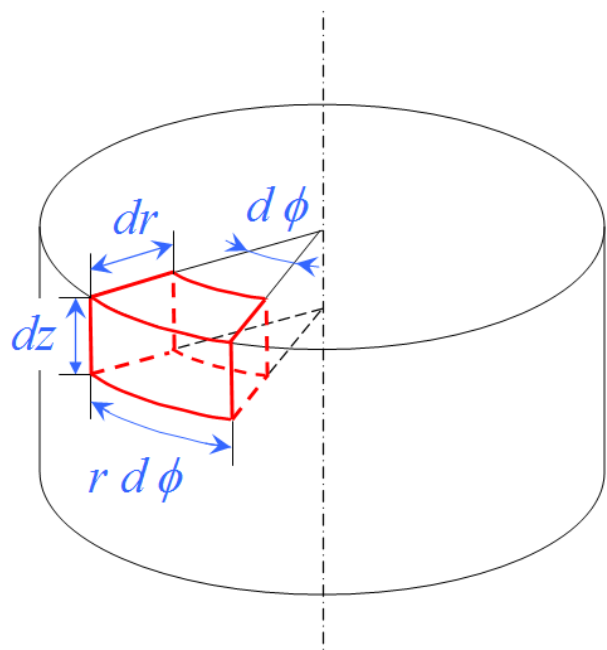
Change of variables in the double integral:

$$\iint_R f dx dy = \iint_R f r dr d\phi$$

(2) Cylindrical polar coordinates (r, ϕ, z)



$$x = r \cos \phi , y = r \sin \phi , z = z$$

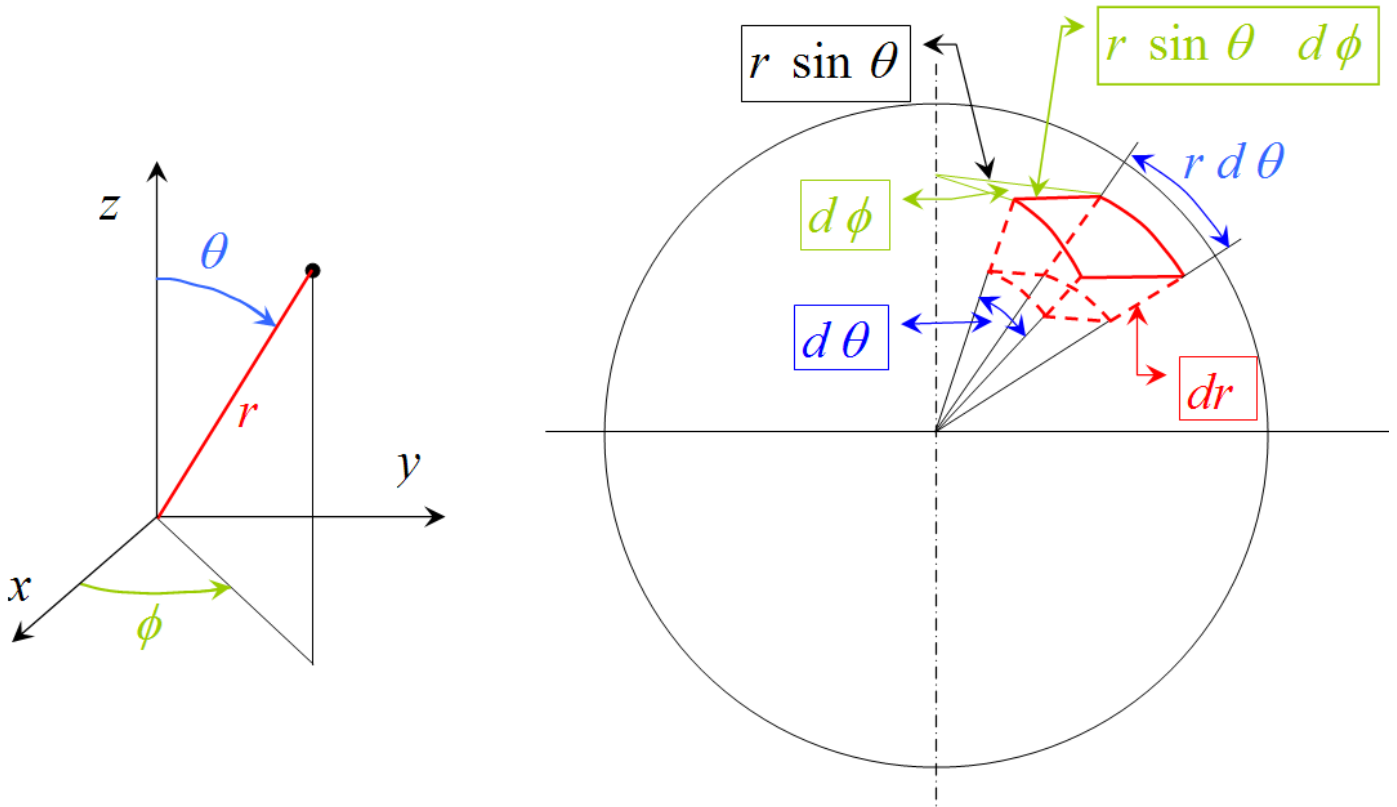


Volume element: $dV = r dr d\phi dz$

Change of variables in the volume (triple) integral:

$$\iiint_V f dx dy dz = \iiint_V f r dr d\phi dz$$

(3) Spherical polar coordinates (r, θ, ϕ)



$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

Volume element:

$$\begin{aligned} dV &= dr \cdot r d\theta \cdot r \sin \theta d\phi \\ &= r^2 \sin \theta dr d\theta d\phi \end{aligned}$$

Change of variables in the volume (triple) integral:

$$\iiint_V f dx dy dz = \iiint_V f r^2 \sin \theta dr d\theta d\phi$$

These changes of variables can also be understood more mathematically, in terms of the mappings of surface or volume elements; see Handout 9.