Curves in Three Dimensions

A single curve: $b:\ t \mapsto (x(t),\ y(t),\ z(t))\ ,\quad t_1 \le t \le t_2\ .$

A tangent vector: $\mathbf{v} = (\dot{x},\ \dot{y},\ \dot{z})\ ,\quad v \equiv |\mathbf{v}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ .$

The unit tangent vector: $\hat{\mathbf{v}} \equiv \dfrac{\mathbf{v}}{v} = \dfrac{(\dot{x},\ \dot{y},\ \dot{z})}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\ .$

For two distinct parametric representations of a given curve, b and b', we have $t' = t'(t)$ and $\mathbf{v}\, dt = \mathbf{v}'\, dt'$. (see Handout 1)

The length of a curve: $S = \int_{t_1}^{t_2} v\, dt = \int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt\ .$

S is independent of the parameterisation. (see Handout 1)

Length along a curve to a point t (the arc length of the curve): $s(t) = \int_{t_1}^{t} v(t')\, dt' = \int_{t_1}^{t} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt'\ .$

A one-parameter family of curves:

$$b_\lambda:\ t \mapsto (x(t,\lambda),\ y(t,\lambda),\ z(t,\lambda))\ ,\quad t_1 \le t \le t_2\ ,\quad \lambda_1 \le \lambda \le \lambda_2\ .$$

A one-parameter family of curves covers a surface embedded into three-dimensional space, so the position of a point belonging to the family is specified using just two parameters, t and λ (so, the geometrical object is two-dimensional), even though each point has three coordinates $(x,\ y,\ z)$ (the space is three-dimensional).