Review of Line Integrals

Consider a curve C in three-dimensional space:

$$C: \quad t \to \Big(x(t),\ y(t),\ z(t)\Big), \quad a \le t \le b\,.$$

Thus, $\mathbf{r}(t) = \Big(x(t),\ y(t),\ z(t)\Big)$ is the position vector of a point on the curve. A tangent to C is given by

$$\mathbf{v} = \dot{\mathbf{r}}(t) = \Big(\dot{x}(t),\ \dot{y}(t),\ \dot{z}(t)\Big)\,,$$

where dot denotes derivative with respect to t.

A line integral of a vector function $\mathbf{F}(\mathbf{r})$ over a curve C is defined by

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \mathbf{F}\Big(\mathbf{r}(t)\Big) \cdot \dot{\mathbf{r}}(t)\, dt \\ &= \int_a^b (F_x\dot{x} + F_y\dot{y} + F_z\dot{z})\, dt\,.\end{aligned}$$

Note that on the right-hand side we have a common definite integral over t. Note also that line integral is a scalar because the integrand is a dot product of two vectors, $\mathbf{F}$ and the tangent vector $\mathbf{v}$ (or $d\mathbf{r}$).

In many cases, it is convenient to take, say, x as a parameter, $t = x$, with $a \le x \le b$. Then, on the curve C, y and z can be considered as functions of x, so that $\dot{x} = \frac{dx}{dx} = 1$, $\dot{y} = \frac{dy}{dx}$, $\dot{z} = \frac{dz}{dx}$, and

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \left(F_x + F_y\frac{dy}{dx} + F_z\frac{dz}{dx}\right) dx$$

Denoting $d\mathbf{r} = (dx,\ dy,\ dz)$, we have $dx = \frac{dx}{dt}dt$, $dy = \frac{dy}{dt}dt$, and $dz = \frac{dz}{dt}dt$, which leads to another useful representation:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b (F_x\, dx + F_y\, dy + F_z\, dz)\,.$$

Example 1 Evaluate $I = \int_C (x^2\, dx + xy\, dy)$ over the following two paths:
(i) $C : t \to (1-t,\ t),\ 0 \le t \le 1,$ and **(ii)** $C : t \to (\cos t,\ \sin t),\ 0 \le t \le \pi/2.$

Solution

(i) Everywhere on the path C, we have $x = 1 - t$ and $y = t$, so that $y = 1 - x$. Thus, C is a segment of a straight line.

Since $\mathbf{F} = (x^2, xy) = \Big((1-t)^2, t(1-t)\Big)$ and $\mathbf{v} = \dot{\mathbf{r}} = (\dot{x},\ \dot{y}) = (-1, 1)$, we have $\mathbf{F}\cdot\mathbf{v} = -(1-t)^2 + t(1-t) = -1 + 2t - t^2 + t - t^2 = -1 + 3t - 2t^2$. Hence,

$$I = \int_0^1 \mathbf{F}\cdot\mathbf{v}\, dt = \int_0^1 (-1 + 3t - 2t^2)\, dt = -1 + \tfrac{3}{2} - \tfrac{2}{3} = -\tfrac{1}{6}\,.$$

The integral can be also found using x as a parameter. Then we take $y = 1 - x$ as an equation defining C, where the starting point is $x = 1$ and the terminal point is $x = 0$ (since $x = 1$ for $t = 0$ and $x = 0$ for $t = 1$). Therefore, the integral over x must be taken from $x = 1$ to $x = 0$. Then $F_x = x^2$, $F_y = xy$. However, $y = 1 - x$ on C, so $F_y = x(1-x)$ and $dy/dx = -1$ on C. Thus,

$$I = \int_1^0 \left(F_x + F_y\frac{dy}{dx}\right) dx = \int_1^0 [x^2 + x(1-x)(-1)]\, dx = \int_1^0 (2x^2 - x)\, dx = \tfrac{2}{3}x^3\Big|_1^0 - \tfrac{1}{2}x^2\Big|_1^0 = -\tfrac{2}{3} + \tfrac{1}{2} = -\tfrac{1}{6}\,.$$

(ii) This path C is a quarter of a circle $x^2+y^2=1$ joining **the same points** $(1,0)$ and $(0,1)$ as in Part (i), but along a different path.

Now $F_x = x^2 = \cos^2 t,\ F_y = xy = \cos t \sin t,\ \dot{x} = -\sin t,\ \dot{y} = \cos t$, so we have

$$\begin{aligned} I &= \int_a^b (F_x \dot{x} + F_y \dot{y})\, dt \\ &= \int_0^{\pi/2} [\cos^2 t(-\sin t) + \cos t \sin t \cos t]\, dt \\ &= \int_0^{\pi/2} (-\cos^2 t \sin t + \cos^2 t \sin t)\, dt \\ &= 0\,. \end{aligned}$$

Conclusion: line integrals generally depend on the path of integration.

1. Line integral over a closed path

If C is a closed path, so that its end-points coincide, $\mathbf{r}(a) = \mathbf{r}(b)$, then the following notation is used:

$$\oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}\ ,$$

and this integral can be evaluated in the same manner as for any other curve. When evaluating a line integral over a closed path, it is important to observe the correct sense of integration along C (this warning actually applies to any line integral).

Example 2 Find $I = \oint_C [(x+y)\, dx + (x-y)\, dy]$ where C is a circle $(x-1)^2 + (y-1)^2 = 4$ traversed in the **clockwise** direction.

Solution

It is convenient to introduce a parametric representation of C. For a circle of a radius R centred at the origin, a convenient (but not unique!) parametric representation is $x = R\cos t,\ y = R\sin t$. Here the end-points, introduced via $a \le t \le b$, are $a = 0,\ b = 2\pi$ if the circle is traversed in the counter-clockwise direction and $a = 2\pi,\ b = 0$ if the circle is traversed in the clockwise direction.

For a circle of a radius 2 centred at the point $(1,1)$, a parametric representation is $x = 1 + 2\cos t$, $y = 1 + 2\sin t$. Here t varies from 2π to 0 because C is traversed in the clockwise direction. Thus, we have

$$I = \int_{2\pi}^{0} \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} \right) dt\ ,$$

where $F_x = x + y = 2 + 2\cos t + 2\sin t$ and $F_y = x - y = 2\cos t - 2\sin t$. Since $dx/dt = -2\sin t$ and $dy/dt = 2\cos t$, we obtain

$$\begin{aligned} I &= \int_{2\pi}^{0} [2(1+\cos t + \sin t)(-2\sin t) + 2(\cos t - \sin t) 2\cos t]\, dt \\ &= \int_{2\pi}^{0} (-4\sin t - 4\sin t\cos t - 4\sin^2 t + 4\cos^2 t - 4\sin t \cos t)\, dt \\ &= \int_{2\pi}^{0} (-4\sin t - 8\sin t \cos t + 4\cos 2t)\, dt \\ &= 4\cos t\Big|_{2\pi}^{0} - 8\int_{2\pi}^{0} \sin t\, d\sin t + 2\sin 2t\Big|_{2\pi}^{0} = 0 - 8\,\tfrac{1}{2}\sin^2 t\Big|_{-2\pi}^{0} + 0 = 0\,. \end{aligned}$$

If the circle were traversed in the counterclockwise direction, the limits of integration over t would be $\int_0^{2\pi} \ldots dt$.

Of course, this example does not imply that any line integral over a closed path is zero. However, $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for an important class of vector functions, namely for **potential fields**, i.e., for $\mathbf{F} = \nabla f$, where f is a scalar field.

2. Line integrals independent of path

Example 3 Find $\int_C [3x^2 y\, dx + (x^3 + 1)\, dy]$ over the following three paths: **(i)** the segment of a straight line joining the points $(0, 0)$ and $(1, 1)$, **(ii)** the segment of a parabola $y = x^2$ joining the same points, and **(iii)** a broken line passing through the points $(0, 0)$, $(1, 0)$ and $(1, 1)$.

Solution

We introduce notation $F_x = 3x^2 y$ and $F_y = x^3 + 1$. We shall use the following expression for a line integral over a path C given by an equation $y = y(x)$:

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \left(F_x + F_y \frac{dy}{dx} \right) dx \ , \tag{1}$$

where $x = a$ and $x = b$ are the starting and terminal points of C.

(i) The first path of integration has an equation $y = x$. Therefore,
$I = \int_0^1 (3x^3 + x^3 + 1)\, dx = \int_0^1 (4x^3 + 1)\, dx = x^4\Big|_0^1 + x\Big|_0^1 = 2.$

(ii) Now we have $y = x^2$ and $I = \int_0^1 [3x^2 \cdot x^2 + (x^3 + 1)2x]\, dx = \int_0^1 (5x^4 + 2x^3)\, dx = x^5\Big|_0^1 + x^3\Big|_0^1 = 2.$

(iii) In this case we subdivide the path of integration into two straight segments, one joining the points $(0, 0)$ and $(0, 1)$, and the other starting at $(0, 1)$ and terminating at $(1, 1)$. We denote these segments as OB and BD, respectively, and then $C = OBD$. (It is useful to make a sketch of the path of integration and to mark clearly its characteristic points.) Thus, we have $I = \int_{C=OBD} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{OB} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} + \int_{BD} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$.

Further, $y = 0$ and $0 \le x \le 1$ along the segment OB and $x = 1$ and $0 \le y \le 1$ along the segment BD. These are the equations describing the paths of integration. Thus,

$$I = \int_0^1 \left(F_x + F_y \frac{dy}{dx} \right) dx + \int_0^1 \left(F_x \frac{dx}{dy} + F_y \right) dy \ ,$$

where in the first integral (taken over OB) the path of integration is given by the equation $y = y(x) = 0$, so that $dy/dx = 0$, whereas in the second integral (taken over BD) the path of integration is specified by the equation $x = x(y) = 1$, so that x and y are interchanged in Eq. (1) and $dx/dy = 0$. In other words, $\int_{(0,0)}^{(1,1)} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{(0,0)}^{(0,1)} F_x\, dx + \int_{(1,0)}^{(1,1)} F_y\, dy$ because $dy = 0$ along the path OB in the first integral and $dx = 0$ along the path BD in the second integral.

We thus have

$$I = \int_0^1 F_x\, dx + \int_0^1 F_y\, dy = \int_0^1 3x^2 \cdot 0\, dx + \int_0^1 (x^3 + 1)\, dy = 0 + 2 = 2 \ .$$

Conclusion: *Some* line integrals do **not** depend on the path of integration.

Example 4 Show that

$$\int_{AB} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A) \qquad \text{if} \quad \mathbf{F}(\mathbf{r}) = \nabla f \ , \tag{2}$$

where AB denotes a curve joining points A and B.

Solution

Let the curve AB be represented in a parametric form, $AB: \ t \to \mathbf{r}(t)$ with the starting point $t = a$ and the end point $t = b$. Then

$$\begin{aligned} \int_{AB} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_a^b \left(F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} \right) dt \\ &= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{df}{dt}\, dt = f\Big(x(t), y(t), z(t)\Big)\Big|_{t=a}^{t=b} \\ &= f(B) - f(A) \ , \end{aligned} \tag{3}$$

where we used the following relation $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$, which follows from the chain rule.

It is worth to remember the simple and convenient formula (2) and to use it wherever possible to evaluate line integrals of potential vector fields.

This formula also shows that the line integral is independent of the path for a potential field, because then the integral is completely determined by the values of the potential f at the end-points. For other fields, the line integral depends on the path of integration (see Example 1).

It is now clear that the line integral of a potential vector field over a closed path vanishes because the starting point coincides with the terminal point ($A = B$), so that Eq. (3) yields $\oint_C \nabla f \cdot d\mathbf{r} = f(A) - f(A) = 0$, where A is an arbitrary point on C.

Example 5 Reconsider Example 3 above to find f such that $\mathbf{F} = \nabla f$ and calculate the integral in a simpler way.

Solution

We have $F_x = 3x^2y$ and $F_y = x^3 + 1$. Thus, condition $\mathbf{F} = \nabla f$ reduces to

$$\frac{\partial f}{\partial x} = 3x^2 y \ , \tag{4}$$

$$\frac{\partial f}{\partial y} = x^3 + 1 \ . \tag{5}$$

Integration of Eq. (4) with respect to x yields $f(x, y) = x^3y + g(y)$, where $g(y)$ is some function of y alone. In order to find out $g(y)$, differentiate this expression with respect to y: $\frac{\partial f}{\partial y} = x^3 + \frac{dg}{dy}$. Comparing this with Eq. (5), we conclude that $dg/dy = 1$, so that $g = y + \text{const}$. Thus, we have $f = y(x^3 + 1) + \text{const}$.

Therefore, $\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = f(B) - f(A) = f(1, 1) - f(0, 0) = 2$ — the same result as obtained in Example 3 above.