Gradient, Divergence and Curl

Question 1 **[5 marks]**

Find the directional derivative of the scalar field $f(x,y,z)=xyz$ in the direction of $\mathbf{b}=(0,4,3)$, at the point $\mathbf{r}=(-2,3,4)$.

Solution

The directional derivative is given by $\partial f/\partial b = \widehat{\mathbf{b}}\cdot\nabla f$.
Here $\mathbf{b}=(0,4,3)$, so $|\mathbf{b}|=\sqrt{0+16+9}=5$, and $\widehat{\mathbf{b}}=\mathbf{b}/|\mathbf{b}|=(0,4/5,3/5)$.
$f(x,y,z)=xyz$, so $\nabla f=(\partial f/\partial x,\partial f/\partial y,\partial f/\partial z)=(yz,xz,xy)$; at point $\mathbf{r}=(-2,3,4)$, $\nabla f=(12,-8,-6)$.
So

$$\frac{\partial f}{\partial b}=\widehat{\mathbf{b}}\cdot\nabla f=(0,\frac{4}{5},\frac{3}{5})\cdot(12,-8,-6)=0+\frac{-32}{5}+\frac{-18}{5}=-10\,.$$

Question 2 **[6+4 marks]**

(a) Expand and simplify $\nabla\cdot\Big(f(r)\mathbf{r}\Big)$, where $r=|\mathbf{r}|$, $\mathbf{r}=(x,y,z)$ and f is a differentiable scalar field.

(b) ($*$) Find $f(r)$ such that $\nabla\cdot\Big(f(r)\mathbf{r}\Big)=0$.

Solution

(a) Since $\mathbf{r}=(x,y,z)$, we have $f(r)\mathbf{r}=(f(r)x,f(r)y,f(r)z)=(xf(r),yf(r),zf(r))$, so

$$\begin{aligned}\nabla\cdot\Big(f(r)\mathbf{r}\Big)&=\frac{\partial}{\partial x}\Big(xf(r)\Big)+\frac{\partial}{\partial y}\Big(yf(r)\Big)+\frac{\partial}{\partial z}\Big(zf(r)\Big)\\&=f(r)+x\frac{df}{dr}\frac{x}{r}+f(r)+y\frac{df}{dr}\frac{y}{r}+f(r)+z\frac{df}{dr}\frac{z}{r}\\&=3f(r)+\frac{df}{dr}\frac{x^2+y^2+z^2}{r}=3f(r)+r\frac{df}{dr}\,.\end{aligned}$$

Here we have used the product rule and chain rule to expand terms such as

$$\frac{\partial}{\partial x}\Big(xf(r)\Big)=f(r)\frac{\partial}{\partial x}(x)+x\frac{\partial}{\partial x}(f(r))=f(r)+x\frac{\partial}{\partial x}(f(r))=f(r)+x\,\frac{df}{dr}\frac{\partial r}{\partial x}\,,$$

and have also used the result that $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$, just as we did in Question 4 of Exercises, Part 3. (Please see those solutions to recall where this result came from, and also for a fuller explanation of the use of the chain rule.)

Alternatively, using the subscript notation,

$$\begin{aligned}\nabla\cdot\Big(f(r)\mathbf{r}\Big)=\frac{\partial}{\partial x_i}\Big(f(r)x_i\Big)&=x_i\frac{\partial f(r)}{\partial x_i}+f(r)\frac{\partial x_i}{\partial x_i}=x_if'(r)\frac{\partial r}{\partial x_i}+3f(r)\\&=\quad f'(r)x_i\frac{x_i}{r}+3f(r)=rf'(r)+3f(r),\end{aligned}$$

where $f'(r)=\dfrac{df}{dr}$. (Remember about summation convention to get the factor 3 right.)

(b) $\nabla\cdot\big(f(r)\mathbf{r}\big) = 0$ if $3f(r)+rf'(r)=0$, where $f'(r)=\dfrac{df}{dr}$. This differential equation can be easily solved by separation of variables:

$$3f+r\frac{df}{dr}=0 \quad\Rightarrow\quad \int\frac{df}{f}=-3\int\frac{dr}{r} \quad\Rightarrow\quad \ln|f|=-3\ln r+C \text{ (note that } r\geq 0) \quad\Rightarrow\quad f=Cr^{-3},$$

where C is an arbitrary constant.

Question 3 **[5 marks]**

Expand and simplify $\nabla\times(r\mathbf{r})$, where $r=|\mathbf{r}|$ and $\mathbf{r}=(x,y,z)$.

Solution

$\nabla\times(r\mathbf{r}) = \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ rx & ry & rz \end{vmatrix}$, so the x-component is given by $\left(\dfrac{\partial}{\partial y}rz-\dfrac{\partial}{\partial z}ry\right) = \left(\dfrac{yz}{r}-\dfrac{yz}{r}\right) = 0$. From the symmetry with respect to cyclic permutations, we conclude that all the remaining components vanish too.

Otherwise, using the subscript notation, $\nabla\times(r\mathbf{r}) = \varepsilon_{ijk}\dfrac{\partial x_k r}{\partial x_j} = \varepsilon_{ijk}\left(\delta_{jk}r + x_k\dfrac{x_j}{r}\right)$ $= \left(\varepsilon_{ijj}r+\dfrac{1}{r}\varepsilon_{ijk}x_jx_k\right) = \mathbf{0}+\dfrac{1}{r}\mathbf{r}\times\mathbf{r}=\mathbf{0}$, since $\varepsilon_{ijj}=\mathbf{0}$ and $\mathbf{u}\times\mathbf{u}=\mathbf{0}$ for any vector $\mathbf{u}$.

Alternatively, using the identity of Question 4

$$\nabla\times(r\mathbf{r}) = \nabla r\times\mathbf{r}+r\nabla\times\mathbf{r} = \frac{1}{r}\mathbf{r}\times\mathbf{r}+\mathbf{0}=\mathbf{0}\ .$$

Note that the vector $\mathbf{r}f(r)$ is **irrotational** for any $f(r)$, i.e., its curl vanishes.

Question 4 **[7 marks]**

Show that $\nabla\times(f\mathbf{v}) = \nabla f\times\mathbf{v}+f\nabla\times\mathbf{v}$, where $f(x,y,z)$ is a differentiable scalar field and $\mathbf{v}(x,y,z)$ is a differentiable vector field.

Solution

We can take advantage of the symmetry with respect to cyclic permutations and only consider the x-component of the identity. Since

$$\nabla\times(f\mathbf{v}) = \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fv_x & fv_y & fv_z \end{vmatrix} = \widehat{\mathbf{x}}\left(\frac{\partial}{\partial y}fv_z-\frac{\partial}{\partial z}fv_y\right)+\widehat{\mathbf{y}}\left(\frac{\partial}{\partial z}fv_x-\frac{\partial}{\partial x}fv_z\right)+\widehat{\mathbf{z}}\left(\frac{\partial}{\partial x}fv_y-\frac{\partial}{\partial y}fv_x\right),$$

the x-component reduces to

$$v_z\frac{\partial f}{\partial y}+f\frac{\partial v_z}{\partial y}-v_y\frac{\partial f}{\partial z}-f\frac{\partial v_y}{\partial z} = \left(v_z\frac{\partial f}{\partial y}-v_y\frac{\partial f}{\partial z}\right)+f\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right). \qquad (1)$$

This cannot be simplified any further. In order to verify that the last expression is indeed equal to $f\times\mathbf{v}+f\nabla\times\mathbf{v}$ (if it is not yet clear), it is useful to write out the x-component of the right-hand side of the identity to be proved:

$$\begin{aligned}[\nabla f\times\mathbf{v}+f\nabla\times\mathbf{v}]_x &= \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}_x + f\begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}_x \\ &= \left(v_z\frac{\partial f}{\partial y}-v_y\frac{\partial f}{\partial z}\right)+f\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right),\end{aligned}$$

which is identical to Eq. (1).

Alternatively, using the subscript notation allows a very compact proof:
$$\nabla \times (f\mathbf{v}) = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(fv_k) = \varepsilon_{ijk}\frac{\partial f}{\partial x_j}v_k + \varepsilon_{ijk}f\frac{\partial v_k}{\partial x_j} = \nabla f \times \mathbf{v} + f\nabla \times \mathbf{v}\ .$$

Question 5 **[3+3+5 marks]**

Given that $\mathbf{c} = (c_x, c_y, c_z)$ is a constant vector, and $\mathbf{r} = (x, y, z)$ is the position vector, simplify the following expressions, expressing your answer in terms of $\mathbf{c}$ wherever possible:

(a) $\nabla(\mathbf{c}\cdot\mathbf{r})$; **(b)** $\mathbf{c}(\nabla\cdot\mathbf{r})$; **(c)** (∗) $\nabla\times(\mathbf{c}\times\mathbf{r})$.

Solution

(a) Here $\mathbf{c}\cdot\mathbf{r}$ is a scalar (the dot product, $\mathbf{c}\cdot\mathbf{r} = c_x x + c_y y + c_z z$), and the expression required is the gradient of this quantity:
$$\nabla(\mathbf{c}\cdot\mathbf{r}) = \nabla(c_x x + c_y y + c_z z)$$
$$= \left(\frac{\partial}{\partial x}(c_x x + c_y y + c_z z)\,,\ \frac{\partial}{\partial y}(c_x x + c_y y + c_z z)\,,\ \frac{\partial}{\partial z}(c_x x + c_y y + c_z z)\right)$$
$$= (c_x, c_y, c_z) = \mathbf{c}\ .$$

(Note that, since $\mathbf{c}$ is a *constant* vector, all derivatives of any of its components vanish.)

(b) Here $\nabla\cdot\mathbf{r}$ is a scalar quantity (the divergence of the position vector $\mathbf{r}$), and the expression required is just this quantity times the vector $\mathbf{c}$:
$$\mathbf{c}(\nabla\cdot\mathbf{r}) = \mathbf{c}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = \mathbf{c}(1+1+1) = 3\,\mathbf{c}\ .$$

(c) Here $\mathbf{c}\times\mathbf{r}$ is a vector quantity (the cross product of the two vectors involved), and the expression required is the curl of this vector product:
$$\mathbf{c}\times\mathbf{r} = \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ c_x & c_y & c_z \\ x & y & z \end{vmatrix} = (\,c_y z - c_z y\,,\ c_z x - c_x z\,,\ c_x y - c_y x\,)\ .$$

So
$$\nabla\times(\mathbf{c}\times\mathbf{r}) = \begin{vmatrix} \widehat{\mathbf{x}} & \widehat{\mathbf{y}} & \widehat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (c_y z - c_z y) & (c_z x - c_x z) & (c_x y - c_y x) \end{vmatrix} = (\,c_x + c_x\,,\ c_y + c_y\,,\ c_z + c_z\,) = 2\mathbf{c}\ .$$

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